Split Exact Sequence

Let R be a ring and $0 \to A_1 \xrightarrow{f} B \xrightarrow{g} A_2 \to 0$ a short exact sequence of R-module homomorphisms. Then the following conditions are equivalent.

(i) There is an R-module homomorphism $h : A_2 \to B$ with $gh = 1_{A_2}$;

(ii) There is an R-module homomorphism $k : B \to A_1$ with $kf = 1_{A_1}$;

(iii) The given sequence is isomorphic (with identity maps on A_1 and A_2) to the direct sum short exact sequence

$$0 \to A_1 \xrightarrow{i_1} A_1 \oplus A_2 \xrightarrow{\pi_2} A_2 \to 0;$$

in particular $B \cong A_1 \oplus A_2$.

A short exact sequence that satisfies the equivalent conditions above is said to be **split** or a **split exact** sequence.

Proof

(Hungerford pg 177)

(i) $\implies$ (iii): The homomorphisms f and h induce a module homomorphism $\varphi : A_1 \oplus A_2 \to B$, given by

$$\varphi(a_1, a_2) = f(a_1) + h(a_2).$$

Verify that the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_1 & \xrightarrow{i_1} & A_1 \oplus A_2 & \xrightarrow{\pi_2} & A_2 & \longrightarrow & 0 \\
 & & \big\downarrow 1_{A_1} & & \big\downarrow \varphi & & \big\downarrow 1_{A_2} & & \\
0 & \longrightarrow & A_1 & \xrightarrow{f} & B & \xrightarrow{g} & A_2 & \longrightarrow & 0
\end{array}$$

is commutative:

$$\varphi i_1(a_1) = f(a_1) = f 1_{A_1}(a_1)$$
$$1_{A_2}\pi_2(a_1, a_2) = a_2 = g\varphi(a_1, a_2) = gh(a_2).$$

By the Short Five Lemma, φ is an isomorphism.

(ii) $\Longrightarrow$ (iii): Verify that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_1 & \xrightarrow{f} & B & \xrightarrow{g} & A_2 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1_{A_1}} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle 1_{A_2}} & & \\ 0 & \longrightarrow & A_1 & \xrightarrow{i_1} & A_1 \oplus A_2 & \xrightarrow{\pi_2} & A_2 & \longrightarrow & 0 \end{array}$$

is commutative, where ψ is the module homomorphism given by $\psi(b) = (k(b), g(b))$:

$$\begin{aligned} \psi f(a_1) &= (kf(a_1), gf(a_1)) \\ &= (a_1, 0) \\ &= i_1 1_{A_1}(a_1) \end{aligned}$$

$$1_{A_2} g(b) = g(b) = \pi_2 \psi(b).$$

Hence the Short Five Lemma implies ψ is an isomorphism.

(iii) $\Longrightarrow$ (i),(ii): Given a commutative diagram with exact rows and φ an isomorphism:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_1 & \underset{\pi_1}{\overset{i_1}{\rightleftarrows}} & A_1 \oplus A_2 & \underset{i_2}{\overset{\pi_2}{\rightleftarrows}} & A_2 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1_{A_1}} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle 1_{A_2}} & & \\ 0 & \longrightarrow & A_1 & \xrightarrow{f} & B & \xrightarrow{g} & A_2 & \longrightarrow & 0 \end{array}$$

define $h : A_2 \to B$ to be φi_2 and $k : B \to A_1$ to be $\pi_1 \varphi^{-1}$.

Then

$$gh(a_2) = g\varphi(0, a_2) = 1_{A_2} \pi_2(0, a_2) = a_2$$

so that $gh = 1_{A_2}$.

$$\begin{aligned} kf(a_1) &= \pi_1 \varphi^{-1} f(a_1) \\ &= \pi_1 \varphi^{-1} f 1_{A_1}(a_1) \\ &= \pi_1 \varphi^{-1} \varphi i_1(a_1) \\ &= \pi_1 1_{A_1 \oplus A_2}(a_1, 0) \\ &= a_1 \end{aligned}$$

so that $kf = 1_{A_1}$.