Analysis Part 7

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Book: Measure and Integral by Wheeden and Zygmund

9 Chapter 9

9.1 Q4

(a)

We will prove by induction that

$$h^{(n)}(x) = \begin{cases} p_n(x^{-1})e^{-x^{-2}} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

where $p_n(x^{-1})$ is a polynomial in x^{-1}.

(Base case: $n = 1$.)

If $x < 0$, it is clear that $h'(x) = 0$.

If $x > 0$, $h'(x) = \frac{d}{dx}(e^{-x^{-2}}) = 2x^{-3}e^{-x^{-2}}$, where $p_1(x^{-1}) = 2x^{-3}$.

If $x = 0$, the left derivative $h'_{-}(0) = 0$ and the right derivative

$$\begin{aligned} h'_{+}(0) &= \lim_{x\to 0^+} \frac{h(x) - h(0)}{x - 0} \\ &= \lim_{x\to 0^+} \frac{e^{-x^{-2}}}{x} \\ &= \lim_{x\to 0^+} \frac{x^{-1}}{e^{x^{-2}}} \\ &= \lim_{x\to 0^+} \frac{-x^{-2}}{-2x^{-3}e^{x^{-2}}} && \text{(L'Hopital's Rule)} \\ &= \lim_{x\to 0^+} \frac{x}{2e^{x^{-2}}} \\ &= 0. \end{aligned}$$

(Inductive Step) Assume that for some $k \geq 1$, $h^{(k)}(x) = p_k(x^{-1})e^{-x^{-2}}$ if $x > 0$, 0 otherwise.

If $x > 0$, using Product Rule, we can see that $h^{(k+1)}(x) = p_{k+1}(x^{-1})e^{-x^{-2}}$ for some polynomial p_{k+1}.

If $x < 0$, again clearly $h^{(k+1)}(x) = 0$.

If $x = 0$, $h^{(k+1)}_{-}(x) = 0$ while the right derivative

$$h^{(k+1)}_{+}(x) = \lim_{x\to 0^+} \frac{p_k(x^{-1})e^{-x^{-2}}}{x}.$$

Note that $p_k(x^{-1}) = \frac{g(x)}{x^m}$ for some polynomial $g(x)$ and some $m \in \mathbb{N}$.

So

$$h^{(k+1)}_{+}(x) = \left(\lim_{x\to 0^+} g(x)\right)\left(\lim_{x\to 0^+} \frac{e^{-x^{-2}}}{x^{m+1}}\right) = 0$$

since

$$\lim_{x\to 0^+} \frac{e^{-x^{-2}}}{x^{m+1}} = \lim_{x\to 0^+} \frac{x^{-m-1}}{e^{x^{-2}}} = 0$$

by repeated application of L'Hopital's Rule.

Thus, by Mathematical Induction, h is in C^∞.

(b)

Let $\phi(x) = x - a$. Clearly ϕ is C^∞. Then $h(x-a) = h(\phi(x))$ is C^∞ since it is the composition of h and ϕ. Similarly, $h(b-x)$ is C^∞. Then $g(x) = h(x-a)h(b-x)$ is C^∞ (we can see this by repeated usage of product rule, or the general Leibniz rule).

If $x \leq a$, then $x - a \leq 0$ so that $h(x-a) = 0$. If $x \geq b$, then $b - x \leq 0$ so that $h(b-x) = 0$. If $a < x < b$, then $x - a > 0$ and $b - x > 0$ so that $g(x) > 0$.

So $\operatorname{supp}(g) = \overline{(a,b)} = [a,b]$.

(c)

Support is a ball:

Define $g(x_1, \ldots, x_n) = h(r^2 - (\sum_{i=1}^n x_i^2))$, where $r > 0$.

Then $g \in C_0^\infty(\mathbb{R}^n)$ and $\operatorname{supp}(g) = \overline{B_r(0)}$, the (closed) ball with radius r centered at the origin.

Support is an interval:

Define $g(x_1, \ldots, x_n) = \prod_{i=1}^n [h(x_i - a_i)h(b_i - x_i)]$ for $a_i < b_i$.

Then $g \in C_0^\infty(\mathbb{R}^n)$ and $\operatorname{supp}(g) = \prod_{i=1}^n [a_i, b_i]$.

9.2 Q5

Lemma 9.2.1. We can choose an open G_2 such that $\overline{G_1} \subset G_2$, and $\overline{G_2} \subset G$.

Proof. We use the fact that $\mathbb{R}^n$ is a normal space: every two disjoint closed sets of $\mathbb{R}^n$ have disjoint open neighborhoods.

Note that $\overline{G_1}$ and G^c (complement of G) are disjoint closed sets. Thus there are disjoint open sets G_2, G_3 such that $\overline{G_1} \subset G_2$ and $G^c \subset G_3$. Note that $G_2 \cap G_3 = \varnothing$ implies $G_2 \subset G_3^c$ and $G^c \subset G_3$ implies $G_3^c \subset G$. Furthermore G_3^c is closed. Then

$$\overline{G_1} \subset G_2 \subset \overline{G_2} \subset G_3^c \subset G.$$

$\square$

Define

$$\begin{aligned}
\epsilon_1 &= \inf\{|x-y| : x \in G_1, y \in G_2^c\} = \operatorname{dist}(G_1, \partial G_2) \\
\epsilon_2 &= \inf\{|x-y| : x \in G_2, y \in G^c\} = \operatorname{dist}(G_2, \partial G)
\end{aligned}$$

and let $\epsilon = \min\{\epsilon_1, \epsilon_2\}$.

By Question 4(c), we can choose $K \in C_0^\infty$ with $\operatorname{supp}(K) = \overline{B_\epsilon(0)}$. Since K is continuous with compact support, K is integrable. By multiplying with a suitable constant, we can further assume that $\int K = 1$.

Let $h(x) = (\chi_{G_2} * K)(x)$. Theorem 9.3 tells us that $h \in C^\infty$, since $\chi_{G_2} \in L^1$ as G_2 is a bounded set.

Let $x \in G_1$. Then

$$\begin{aligned}
h(x) &= \int_{\mathbb{R}^n} \chi_{G_2}(x-t)K(t)\,dt \\
&= \int_{B_\epsilon(0)} \chi_{G_2}(x-t)K(t)\,dt && \text{(since } \operatorname{supp}(K) = \overline{B_\epsilon(0)}\text{)} \\
&= \int_{B_\epsilon(0)} K(t)\,dt && \text{(since } x-t \in G_2 \text{ for } |t| < \epsilon\text{)} \\
&= 1.
\end{aligned}$$

Finally, if $x \in G^c$, then

$$\begin{aligned}
h(x) &= \int_{\mathbb{R}^n} \chi_{G_2}(x-t)K(t)\,dt \\
&= \int_{B_\epsilon(0)} \chi_{G_2}(x-t)K(t)\,dt \\
&= 0
\end{aligned}$$

since $x - t \notin G_2$ for $|t| < \epsilon$.

9.3 Q6

Let $K(x) \leq M$ for all $x \in \mathbb{R}^n$.

Then

$$\begin{aligned}
|(f * K)(x)| &= \left| \int_{\mathbb{R}^n} f(x-t)K(t)\,dt \right| \\
&\leq \left| \int_{\mathbb{R}^n} f(x-t)\,dt \right| M \\
&\leq \|f\|_1 M \\
&< \infty.
\end{aligned}$$

So $f * K$ is bounded.

Let $\epsilon > 0$. Since K is uniformly continuous on $\mathbb{R}^n$, there exists $\delta(\epsilon) > 0$ such that for any $x, y \in \mathbb{R}^n$, if $|x-y| < \delta$, then $|K(x) - K(y)| < \frac{\epsilon}{\|f\|_1 + 1}$.

Then for all $|x - y| < \delta$,

$$\begin{aligned}
|(f * K)(x) - (f * K)(y)| &= \left| \int_{\mathbb{R}^n} f(t)K(x-t)\,dt - \int_{\mathbb{R}^n} f(t)K(y-t)\,dt \right| \\
&= \left| \int_{\mathbb{R}^n} f(t)[K(x-t) - K(y-t)]\,dt \right| \\
&\leq \frac{\epsilon}{\|f\|_1 + 1} \int_{\mathbb{R}^n} |f(t)|\,dt \\
&= \frac{\epsilon}{\|f\|_1 + 1} \|f\|_1 \\
&< \epsilon.
\end{aligned}$$

Hence $f * K$ is uniformly continuous.

9.4 Q7

It is shown in the textbook that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) P_y(x) = 0 \qquad \text{for } y > 0.$$

Thus it suffices to show that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) f(x,y)=\int_{-\infty}^{\infty} f(t)\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) P_y(x-t)\,dt=0.$$

Lemma 9.4.1. $\frac{\partial}{\partial x} f(x,y)=\int_{-\infty}^{\infty} f(t)\frac{\partial}{\partial x}P_y(x-t)\,dt.$

Proof. Let $y>0$. By definition,

$$\frac{\partial}{\partial x}P_y(x-t)=\lim_{h\to 0}\frac{P_y(x+h-t)-P_y(x-t)}{h}.$$

Let (h_n) be a sequence tending to 0, $h_n\neq 0$, and define

$$\phi_n(x,t)=\frac{P_y(x+h_n-t)-P_y(x-t)}{h_n}.$$

It follows that $\frac{\partial}{\partial x}P_y(x-t)=\lim_{n\to\infty}\phi_n(x,t)$.

Using Mean Value Theorem, we have

$$\begin{aligned}|\phi_n(x,t)|&=\left|\frac{\partial}{\partial x}P_y(c-t)\right| \qquad \text{for some } c\in(x,x+h_n)\\ &\le \sup_{x\in\mathbb{R}}\left|\frac{\partial}{\partial x}P_y(x-t)\right|.\end{aligned}$$

Let $R>0$ be arbitrary. For $|x|\le R$, note that

$$\begin{aligned}\left|\frac{\partial}{\partial x}P_y(x-t)\right|&=\frac{1}{\pi}\left|\frac{\partial}{\partial x}\frac{y}{y^2+(x-t)^2}\right|\\ &=\frac{1}{\pi}\left|\frac{1}{y^2+(x-t)^2}\cdot\frac{-2y(x-t)}{y^2+(x-t)^2}\right| \qquad \text{(Quotient Rule)}\\ &\le\frac{1}{\pi}\left(\frac{1}{y^2+(x-t)^2}\right)\\ &\qquad \left(\text{since } \left|\tfrac{2y(x-t)}{y^2+(x-t)^2}\right|\le 1 \text{ by the inequality } 2|ab|\le a^2+b^2\right)\\ &\le\frac{1}{\pi}\frac{1}{y^2+(\max\{|t|-R,0\})^2}:=g(t).\end{aligned}$$

Note that if R is sufficiently large, for $|x|>R$, $|\frac{\partial}{\partial x}P_y(x-t)|$ is arbitrarily small since $\frac{1}{\pi}(\frac{1}{y^2+(x-t)^2})$ decays to 0 as $|x|\to\infty$.

Note that $g \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Next, note that $g \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ implies $g \in L^q(\mathbb{R})$, where q is the Hölder conjugate of p. Explanation: $\int |g|^q \leq \|g\|_\infty^{q-1} \int |g| < \infty$.

Hence $\|fg\|_1 \leq \|f\|_p \|g\|_q < \infty$ by Hölder's inequality. Hence $|f(t)\phi_n(x,t)| \leq |f(t)g(t)|$ where fg is integrable. Thus, by Lebesgue's DCT,

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(t)\phi_n(x,t)\,dt = \int_{-\infty}^{\infty} f(t) \lim_{n\to\infty} \phi_n(x,t)\,dt.$$

Since $y > 0$ and $R > 0$ are arbitrary,

$$\frac{\partial}{\partial x} f(x,y) = \frac{\partial}{\partial x} \int_{-\infty}^{\infty} f(y) P_y(x-t)\,dt = \int_{-\infty}^{\infty} f(t) \frac{\partial}{\partial x} P_y(x-t)\,dt$$

holds for the upper half-plane. □

Note that the main argument above is to find an integrable function that dominates $|f(t)\frac{\partial}{\partial x} P_y(x-t)|$.

Similarly, we can prove $\frac{\partial^2}{\partial x^2} f(x,y) = \int_{-\infty}^{\infty} f(t) \frac{\partial^2}{\partial x^2} P_y(x-t)\,dt$ and the analogous statements for $\frac{\partial}{\partial y} f(x,y)$ and $\frac{\partial^2}{\partial y^2} f(x,y)$.

Briefly, since the Poisson kernel is smooth, all derivatives of it are bounded on all compact subsets of the upper half-plane. Furthermore, it decays to zero as $|x| \to \infty$, with faster decay for higher-order derivatives. Thus our dominating function $g(t)$ (multiplied by a constant) works for all derivatives.

9.5 Q8

We note that for $s > 0$,

$$K(s,t) = K(s \cdot 1, s \cdot t/s) = s^{-1} K(1, t/s).$$

Therefore

$$\begin{aligned}(Tf)(s) &= \int_0^\infty f(t)s^{-1}K(1,t/s)\,dt \\ &= \int_0^\infty f(ts)s^{-1}K(1,t)s\,dt \\ &= \int_0^\infty f(ts)K(1,t)\,dt.\end{aligned}$$

(Case: $1 \leq p < \infty$.) Then,

$$\begin{aligned}\|Tf\|_p &= \left(\int \left|\int_0^\infty f(ts)K(1,t)\,dt\right|^p ds\right)^{1/p} \\ &\leq \left(\int \left(\int_0^\infty |f(ts)K(1,t)|\;dt\right)^p ds\right)^{1/p} \\ &\leq \int_0^\infty \left(\int |f(ts)K(1,t)|^p\,ds\right)^{1/p} dt \\ &\qquad\qquad\qquad\qquad \text{(by Minkowski's integral inequality)} \\ &= \int_0^\infty \left(\int |f(ts)|^p\,ds\right)^{1/p} K(1,t)\,dt \\ &= \int_0^\infty \left(\int t^{-1}|f(s)|^p\,ds\right)^{1/p} K(1,t)\,dt \\ &= \|f\|_p \int_0^\infty t^{-1/p}K(1,t)\,dt \\ &= \gamma\|f\|_p.\end{aligned}$$

(Case: $p = \infty$.)

$$\begin{aligned}\|Tf\|_\infty &= \operatorname{ess\,sup}\left|\int_0^\infty f(ts)K(1,t)\,dt\right| \\ &\leq \|f\|_\infty \left|\int_0^\infty K(1,t)\,dt\right| \\ &= \gamma\|f\|_\infty. \qquad\qquad \left(\text{since } \textstyle\int_0^\infty K(1,t)\,dt = \gamma \text{ for } p = \infty\right)\end{aligned}$$

9.6 Q11

(Case: $\gamma \neq 0$).

Consider $g(x) = K(x)/\gamma$. Then $g \in L^1(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} g = 1$. Let $g_\epsilon(x) = \epsilon^{-n} g(\frac{x}{\epsilon})$.

Then

$$\begin{aligned} \|f_\epsilon - \gamma f\|_p &= \|f * K_\epsilon - \gamma f\|_p \\ &= |\gamma| \|f * g_\epsilon - f\|_p \\ &\to 0 \qquad \text{as } \epsilon \to 0. \qquad\qquad \text{(By Theorem 9.6)} \end{aligned}$$

(Case: $\gamma = 0$).

Let $h \in L^1(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} h = 1$. Then $K + h \in L^1(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n}(K + h) = 1$. Let $h_\epsilon(x) = \epsilon^{-n} h(\frac{x}{\epsilon})$.

$$\begin{aligned} \|f_\epsilon - \gamma f\|_p &= \|f * K_\epsilon\|_p \\ &= \|f * (K_\epsilon + h_\epsilon) - f + f - f * h_\epsilon\|_p \\ &\leq \|f * (K + h)_\epsilon - f\|_p + \|f * h_\epsilon - f\|_p \\ &\to 0 \qquad \text{as } \epsilon \to 0. \qquad\qquad \text{(By Theorem 9.6)} \end{aligned}$$

Analogous results for Theorems 9.8, 9.9 and 9.13 can be obtained by replacing $f_\epsilon \to f$ by $f_\epsilon \to \gamma f$ as $\epsilon \to 0$. The proof is similar to that of the generalized Theorem 9.6.

9.7 Q13

Note that $|I_{k,j}|^{-1} = 2^k$, and

$$f_k(x) = \sum_{j=1}^{2^k} 2^k \int_{I_{k,j}} f(t)\, dt \chi_{I_{k,j}}(x).$$

Lemma 9.7.1. $f_k \in L^p(0, 1)$ for all $k \in \mathbb{N}$.

Proof.

$$
\begin{aligned}
\int_0^1 |f_k(x)|^p \, dx &= \int_0^1 \left| \sum_{j=1}^{2^k} 2^k \int_{I_{k,j}} f(t)\, dt \chi_{I_{k,j}}(x) \right|^p \, dx \\
&= 2^{kp} \sum_{j=1}^{2^k} \int_{I_{k,j}} \left| \int_{I_{k,j}} f(t)\, dt \right|^p \, dx \\
&\leq 2^{kp} \sum_{j=1}^{2^k} \int_{I_{k,j}} \left[\int_{I_{k,j}} |f(t)|\, dt \right]^p \, dx \\
&\leq 2^{kp} \sum_{j=1}^{2^k} \int_{I_{k,j}} \left[\left(\int_{I_{k,j}} |f(t)|^p \, dt \right)^{1/p} \left(\int_{I_{k,j}} |1|^{p'} \, dt \right)^{1/p'} \right]^p \, dx \\
&\qquad\qquad \text{(By Hölder's inequality)} \\
&= 2^{kp} \sum_{j=1}^{2^k} \int_{I_{k,j}} \left[\left(\int_{I_{k,j}} |f(t)|^p \, dt \right) 2^{-kp/p'} \right] \, dx \\
&= 2^{kp-k-kp/p'} \sum_{j=1}^{2^k} \int_{I_{k,j}} |f(t)|^p \, dt \\
&= \int_0^1 |f(t)|^p \, dt \\
&= \|f\|_p^p < \infty.
\end{aligned}
$$

□

By Lebesgue's Differentiation Theorem, $f_k \to f$ a.e. By Fatou's Lemma, we have

$$
\begin{aligned}
\int_0^1 |f|^p &\leq \liminf \int_0^1 |f_k|^p \\
&\leq \limsup \int_0^1 |f_k|^p \\
&\leq \int_0^1 |f|^p. \qquad \text{(since } \textstyle\int_0^1 |f_k|^p \leq \int_0^1 |f|^p \text{ for all } k\text{)}
\end{aligned}
$$

Hence $\|f_k\|_p \to \|f\|_p$.

Using Chapter 8 Exercise 12 (If $f_k \to f$ a.e. and $\|f_k\|_p \to \|f\|_p$, $0 < p < \infty$, then $\|f - f_k\|_p \to 0$), we can conclude that $f_k \to f$ in $L^p(0, 1)$ norm.