Analysis Part 6

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Book: Measure and Integral by Wheeden and Zygmund

8 Chapter 8

8.1 Q2

For $\|g\|_{p'} \leq 1$,

$$\|f\|_p \geq \|f\|_p \|g\|_{p'} \geq \int_E |fg| \geq \int_E fg$$

using Hölder's inequality. Therefore $\|f\|_p \geq \sup \int_E fg$. So we only need to prove the opposite inequality.

(Case: $p = 1$**).** Let $g(x) = \operatorname{sgn}(f(x))$, where $\operatorname{sgn}(x) = \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$

Then $\|g\|_\infty \leq 1$ and $\int_E fg = \int_E |f|$ exists[1].

Hence $\sup \int_E fg \geq \int_E |f| = \|f\|_1$.

[1]By Theorem 5.1: Let f be a nonnegative function defined on a measurable set E. Then $\int_E f$ exists iff f is measurable.

(Case: $p = \infty$**).**

(Subcase: $\|f\|_\infty = 0$**).** If $\|f\|_\infty = 0$, then $f = 0$ a.e. in E, so $\|f\|_\infty = \sup \int_E fg = 0$.

(Subcase: $0 < \|f\|_\infty < \infty$**).** If $0 < \|f\|_\infty < \infty$, we may assume without loss of generality that $\|f\|_\infty = 1$.

Define $E_k = \{x \in E : |f(x)| > 1 - \frac{1}{k}\}$ for each $k \in \mathbb{N}$. Note that since $\|f\|_\infty = 1$, $|E_k| > 0$ for each k.

Define

$$g_k = \frac{1}{|A_k|} \chi_{A_k} \text{sgn}(f),$$

where A_k is a measurable subset of E_k such that $0 < |A_k| < \infty$. Such an A_k exists by considering the intersection of E_k with a ball of large enough radius, i.e. $A_k = E_k \cap B_N(0)$ for some N. Then,

$$\|g_k\|_1 = \int_E |g_k| = \int_{A_k} \frac{1}{|A_k|} |\text{sgn}(f)| \leq \int_{A_k} \frac{1}{|A_k|} = 1$$

and $\int_E fg_k = \int_{A_k} \frac{|f|}{|A_k|}$ exists.

Note that

$$\int_E fg_k = \frac{1}{|A_k|} \int_{A_k} |f| \geq \frac{1}{|A_k|} \int_{A_k} (1 - \frac{1}{k}) = 1 - \frac{1}{k}$$

for all k. Thus $\sup \int_E fg \geq \int_E fg_k \geq 1 - \frac{1}{k}$ for all $k \in \mathbb{N}$ which implies $\sup \int_E fg \geq 1 = \|f|_\infty$.

(Subcase: $\|f\|_\infty = \infty$**).** Define $E_k = \{x \in E : |f(x)| > k\}$ for $k \in \mathbb{N}$. Since $\|f\|_\infty = \infty$, $|E_k| > 0$ for all k. Similarly, define $g_k = \frac{1}{|A_k|} \chi_{A_k} \text{sgn}(f)$, where $A_k \subseteq E_k$ and $0 < |A_k| < \infty$. Then, $\|g_k\|_1 \leq 1$ as before and $\int_E fg_k$ exists. Note that

$$\int_E fg_k = \frac{1}{|A_k|} \int_{A_k} |f| \geq \frac{1}{|A_k|} \int_{A_k} k = k$$

for all k. Thus $\sup \int_E fg \geq \int_E fg_k \geq k$ for all $k \in \mathbb{N}$ which implies

$$\sup \int_E fg = \infty = \|f\|_\infty.$$

Part 2: Show also that for $1 \le p \le \infty$, a real-valued measurable f belongs to $L^p(E)$ if $fg \in L^1(E)$ for every $g \in L^{p'}(E)$, $\frac{1}{p} + \frac{1}{p'} = 1$.

Lemma 8.1.1. There exists $M > 0$ such that $\|fg\|_1 \le M$, for all $g \in L^{p'}(E)$, $\|g\|_{p'} \le 1$.

Proof. Suppose not. Then we have a sequence of $L^{p'}$ functions $\{g_k\}$ with $\|g_k\|_{p'} \le 1$ where $\int_E |fg_k| > 3^k$. Let $g = \sum_{k=1}^{\infty} 2^{-k}|g_k|$. Then

$$\|g\|_{p'} \le \sum_{k=1}^{\infty} 2^{-k} = 1$$

but

$$\int_E |fg| = \sum_{k=1}^{\infty} 2^{-k} \int_E |fg_k| > \sum_{k=1}^{\infty} (\frac{3}{2})^k = \infty$$

so $fg \notin L^1(E)$. This is a contradiction. □

So

$$\|f\|_p = \sup_{\|g\|_{p'} \le 1} \int_E fg \le \sup_{\|g\|_{p'} \le 1} \|fg\|_1 \le M < \infty.$$

Thus $f \in L^p(E)$.

Part 3: Show if $f \notin L^p(E)$, then there exists $g \in L^{p'}(E)$ such that $fg \notin L^1(E)$.

The contrapositive of the above is: If for all $g \in L^{p'}(E)$, $fg \in L^1(E)$, then $f \in L^p(E)$. This is exactly what we proved in Part 2.

8.2 Q8

(Case: $p = 1$).

$$\iint |f(x,y)|\, dx\, dy = \iint |f(x,y)|\, dy\, dx$$

by Tonelli's Theorem.

(Case: $1 < p < \infty$**)**.

$$\int \left[\int |f(x,y)|\,dx \right]^p dy$$
$$= \iint \left[\int |f(z,y)|\,dz \right]^{p-1} |f(x,y)|\,dx\,dy$$
$$= \iint \left[\int |f(z,y)|\,dz \right]^{p-1} |f(x,y)|\,dy\,dx \quad \text{(Tonelli's Theorem)}$$
$$= \iint |FG|\,dy\,dx \quad \left(\text{where } F = \left[\int |f(z,y)|\,dz \right]^{p-1},\ G = |f(x,y)|\right)$$
$$\leq \int \left(\int |F|^{\frac{p}{p-1}}\,dy \right)^{\frac{p-1}{p}} \left(\int |G|^p\,dy \right)^{\frac{1}{p}} dx$$

(by Hölder's inequality where $p' = \frac{p}{p-1}$)

$$= \int \left(\int \left[\int |f(z,y)|\,dz \right]^p dy \right)^{\frac{p-1}{p}} \left(\int |f(x,y)|^p\,dy \right)^{\frac{1}{p}} dx$$
$$= \left(\int \left[\int |f(z,y)|\,dz \right]^p dy \right)^{\frac{p-1}{p}} \cdot \left(\int \left[\int |f(x,y)|^p\,dy \right]^{\frac{1}{p}} dx \right)$$
$$= \left(\int \left[\int |f(x,y)|\,dx \right]^p dy \right)^{\frac{p-1}{p}} \cdot \left(\int \left[\int |f(x,y)|^p\,dy \right]^{\frac{1}{p}} dx \right).$$

Denote LHS $= \left[\int \left[\int |f(x,y)|\,dx\right]^p\,dy\right]^{\frac{1}{p}}$.

(Subcase: LHS=0). Then $f(x,y) = 0$ a.e. and the inequality is trivial.

(Subcase: $0 <$ **LHS** $< \infty$**)**. Divide both sides (in the inequality we proved above) by $0 < \left(\int \left[\int |f(x,y)|\,dx\right]^p\,dy\right)^{\frac{p-1}{p}} < \infty$ to get

$$\left[\int \left[\int |f(x,y)|\,dx \right]^p dy \right]^{\frac{1}{p}} \leq \int \left[\int |f(x,y)|^p\,dy \right]^{\frac{1}{p}} dx$$

since $1 - \frac{p-1}{p} = \frac{1}{p}$.

(Subcase: LHS $= \infty$**)**. We may assume $|\{f(x,y) = \infty\}| = 0$, otherwise both sides of the inequality will be infinite. Let

$$g_k(x,y) := |f(x,y)| \cdot \chi_{\{|f(x,y)|<k\}}(x,y) \cdot \chi_{\{x^2+y^2<k\}}(x,y).$$

Note that $0 \le g_k(x,y) \nearrow |f(x,y)|$ a.e. Then by the previous subcase we have

$$\left[\int \left[\int |g_k(x,y)|\,dx\right]^p dy\right]^{\frac{1}{p}} \le \int \left[\int |g_k(x,y)|^p\,dy\right]^{\frac{1}{p}} dx$$

for each k. Then taking limits as $k \to \infty$ and using Monotone Convergence Theorem gives

$$\infty = \left[\int \left[\int |f(x,y)|\,dx\right]^p dy\right]^{\frac{1}{p}} \le \int \left[\int |f(x,y)|^p\,dy\right]^{\frac{1}{p}} dx.$$

8.3 Q12

Assume $\|f - f_k\|_p \to 0$.

(Case: $0 < p < 1$**).**

Lemma 8.3.1. If $0 < p < 1$, $|a+b|^p \le |a|^p + |b|^p$ for all $a, b \in \mathbb{R}$.

Proof.

$$1 = \frac{|a|}{|a|+|b|} + \frac{|b|}{|a|+|b|} \le \left(\frac{|a|}{|a|+|b|}\right)^p + \left(\frac{|b|}{|a|+|b|}\right)^p = \frac{|a|^p + |b|^p}{(|a|+|b|)^p}.$$

Hence $|a+b|^p \le (|a|+|b|)^p \le |a|^p + |b|^p$. □

Hence, using $|a|^p \le |a-b|^p + |b|^p$ and $|b|^p \le |a-b|^p + |a|^p$ we see that

$$||a|^p - |b|^p| \le |a-b|^p. \tag{†}$$

Thus

$$\begin{aligned} \left|\|f_k\|_p^p - \|f\|_p^p\right| &= \left|\int (|f_k|^p - |f|^p)\right| \\ &\le \int ||f_k|^p - |f|^p| \\ &\le \int |f_k - f|^p && \text{(using †)} \\ &= \|f - f_k\|_p^p \to 0 \quad \text{as } k \to \infty. \end{aligned}$$

Hence $\|f_k\|_p \to \|f\|_p$.

(Case: $1 \leq p \leq \infty$.) By Minkowski's inequality, $\|f\|_p \leq \|f - f_k\|_p + \|f_k\|_p$ and $\|f_k\|_p \leq \|f - f_k\|_p + \|f\|_p$ so that

$$\left|\|f_k\|_p - \|f\|_p\right| \leq \|f - f_k\|_p \to 0$$

as $k \to \infty$. Done.

(Converse). Assume $f_k \to f$ a.e. and $\|f_k\|_p \to \|f\|_p$, $0 < p < \infty$.

Lemma 8.3.2. For $a, b \in \mathbb{R}$, $|a + b|^p \leq 2^{p-1}(|a|^p + |b|^p)$ for $1 \leq p < \infty$.

Proof. By convexity of $|x|^p$ for $1 \leq p < \infty$,

$$\left|\frac{1}{2}a + \frac{1}{2}b\right|^p \leq \frac{1}{2}|a|^p + \frac{1}{2}|b|^p.$$

Multiplying throughout by 2^p gives

$$|a + b|^p \leq 2^{p-1}(|a|^p + |b|^p).$$

□

Thus together with Lemma 8.3.1, for $0 < p < \infty$ we have $|f - f_k|^p \leq c(|f|^p + |f_k|^p)$ with $c = \max\{2^{p-1}, 1\}$.

Note that $|f - f_k|^p \to 0$ a.e. and $\phi_k := c(|f|^p + |f_k|^p) \to \phi := 2c|f|^p$ a.e. which is integrable. Also, $\int \phi_k \to \int \phi$ since $\|f_k\|_p^p \to \|f\|_p^p$. By Generalized Lebesgue's DCT, we have $\int |f - f_k|^p \to 0$ thus

$$\|f - f_k\|_p \to 0.$$

For completeness we state and prove Generalized Lebesgue's DCT:

Theorem 8.3.3 (Generalized Lebesgue Dominated Convergence Theorem)**.** Let $\{f_k\}$ and $\{\phi_k\}$ be sequences of measurable functions on E satisfying $f_k \to f$ a.e. in E, $\phi_k \to \phi$ a.e. in E, and $|f_k| \leq \phi_k$ a.e. in E. If $\phi \in L(E)$ and $\int_E \phi_k \to \int_E \phi$, then $\int_E |f_k - f| \to 0$.

Proof. We have $|f_k - f| \le |f_k| + |f| \le \phi_k + \phi$. Applying Fatou's lemma to the non-negative sequence

$$h_k = \phi_k + \phi - |f_k - f|,$$

we get

$$2\int_E \phi \le \liminf_{k\to\infty} \int_E (\phi_k + \phi - |f_k - f|).$$

That is,

$$2\int_E \phi \le 2\int_E \phi - \limsup_{k\to\infty} \int_E |f_k - f|.$$

Since $\int_E \phi < \infty$, we get $\limsup_{k\to\infty} \int_E |f_k - f| \le 0$. Since $\liminf_{k\to\infty} \int_E |f_k - f| \ge 0$, this implies $\lim_{k\to\infty} \int_E |f_k - f| = 0$. $\square$

(Show that the converse may fail for $p = \infty$). Consider $f_k = \chi_{[-k,k]} \in L^\infty(\mathbb{R})$. Then $f_k \to f$ a.e. where $f(x) \equiv 1$, and $\|f_k\|_\infty \to \|f\|_\infty = 1$. However $\|f - f_k\|_\infty = 1 \not\to 0$.

8.4 Q13

(Case: $|E| < \infty$, where E is the domain of integration). We may assume $|E| > 0$, $M > 0$, $\|g\|_{p'} > 0$ otherwise the result is trivially true. Also, by Fatou's Lemma,

$$\|f\|_p \le \liminf_{k\to\infty} \|f_k\|_p \le M.$$

Let $\epsilon > 0$. Since $g \in L^{p'}$, so $g^{p'} \in L^1$ and there exists $\delta > 0$ such that for any measurable subset $A \subseteq E$ with $|A| < \delta$, $\int_A |g^{p'}| < \epsilon^{p'}$.

Since $f_k \to f$ a.e. (f is finite a.e. since $f \in L^p$), by Egorov's Theorem there exists closed $F \subseteq E$ such that $|E \setminus F| < \delta$ and $\{f_k\}$ converge uniformly to f on F. That is, there exists $N(\epsilon)$ such that for $k \ge N$, $|f_k(x) - f(x)| < \epsilon$ for all $x \in F$.

Then for $k \geq N$,

$$\begin{aligned}
\left| \int_E f_k g - fg \right| &\leq \int_E |f_k - f||g| \\
&= \int_{E \setminus F} |f_k - f||g| + \int_F |f_k - f||g| \\
&\leq \left(\int_{E \setminus F} |f_k - f|^p \right)^{\frac{1}{p}} \left(\int_{E \setminus F} |g|^{p'} \right)^{\frac{1}{p'}} + \epsilon \int_F |g| \\
&\qquad \text{(by Hölder's inequality)} \\
&< \|f_k - f\|_p (\epsilon) + \epsilon \left(\int_F |g|^{p'} \right)^{\frac{1}{p'}} \left(\int_F |1|^p \right)^{\frac{1}{p}} \\
&\qquad \text{(by Hölder's inequality)} \\
&\leq 2M\epsilon + \epsilon \|g\|_{p'} |E|^{\frac{1}{p}} \\
&= \epsilon(2M + \|g\|_{p'} |E|^{\frac{1}{p}}).
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, this means $\int_E f_g \to \int_E fg$.

(Case: $|E| = \infty$**).**

Define $E_N = E \cap B_N(0)$, where $B_N(0)$ is the ball with radius N centered at the origin. Then $|E_N| < \infty$, so there exists $N_1 > 0$ such that for $N \geq N_1$, $\int_{E_N} |f_k - f||g| < \epsilon$.

Since $|g|^{p'} \chi_{E_N} \nearrow |g|^{p'}$ on E, by Monotone Convergence Theorem,

$$\lim_{N \to \infty} \int_{E_N} |g|^{p'} = \int_E |g|^{p'} < \infty.$$

Thus there exists $N_2 > 0$ such that for $N \geq N_2$, $\int_{E \setminus E_N} |g|^{p'} < \epsilon^{p'}$.

Then for $N \geq \max\{N_1, N_2\}$,

$$\begin{aligned}
\int_E |f_k g - fg| &= \int_{E_N} |f_k - f||g| + \int_{E \setminus E_N} |f_k - f||g| \\
&< \epsilon + \left(\int_{E \setminus E_N} |f_k - f|^p \right)^{\frac{1}{p}} \left(\int_{E \setminus E_N} |g|^{p'} \right)^{\frac{1}{p'}} \\
&\qquad \text{(by Hölder's inequality)} \\
&< \epsilon + \|f_k - f\|_p(\epsilon) \\
&\leq \epsilon + 2M\epsilon \\
&= \epsilon(1 + 2M).
\end{aligned}$$

so that $\int_E f_k g \to \int_E fg$.

(Show that the result is false if $p = 1$).

Let $f_k := k\chi_{[0,\frac{1}{k}]}$. Then $f_k \to f$ a.e., where $f \equiv 0$. Note that $\int_{\mathbb{R}} |f_k| = 1$, $\int_{\mathbb{R}} |f| = 0$ so that $f_k, f \in L^1(\mathbb{R})$. Similarly, $\|f_k\|_1 \leq M = 1$.

However if $g \equiv 1 \in L^\infty$, $\int_{\mathbb{R}} f_k g = 1$ for all k but $\int_{\mathbb{R}} fg = 0$.

8.5 Q15

Lemma 8.5.1 (Q14a)**.** Verify that the system

$$\{\frac{1}{2}, \cos x, \sin x, \ldots, \cos kx, \sin kx, \ldots\}$$

is orthogonal on any interval of length 2π.

Proof. Since the functions are all periodic with period 2π, it suffices to verify orthogonality on $[0, 2\pi]$. Using trigonometric factor formulae, check that for

$m, n \geq 1$,

$$\int_0^{2\pi} \frac{1}{2} \cos mx \, dx = \int_0^{2\pi} \frac{1}{2} \sin mx \, dx = 0$$

$$\int_0^{2\pi} \cos mx \cos nx \, dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}$$

$$\int_0^{2\pi} \cos mx \sin nx \, dx = 0$$

$$\int_0^{2\pi} \sin mx \sin nx \, dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n. \end{cases}$$

□

Normalize the previous orthogonal system to obtain the orthonormal system

$$\{\phi_k\} := \{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos x, \frac{1}{\sqrt{\pi}} \sin x, \dots, \frac{1}{\sqrt{\pi}} \cos mx, \frac{1}{\sqrt{\pi}} \sin mx, \dots\}$$

in $L^2(0, 2\pi)$.

By Bessel's inequality,

$$\sum_{k=1}^{\infty} \left| \int_0^{2\pi} f(x)\phi_k(x) \, dx \right|^2 \leq \|f\|_2 < \infty.$$

Hence, $\int_0^{2\pi} f(x)\phi_k(x) \, dx \to 0$ as $k \to \infty$ so that

$$\lim_{k\to\infty} \int_0^{2\pi} f(x) \cos kx \, dx = \lim_{k\to\infty} \int_0^{2\pi} f(x) \sin kx \, dx = 0.$$

(Prove that the same is true if $f \in L^1(0, 2\pi)$**).**

Lemma 8.5.2. If $f \in L^1$ and $\epsilon > 0$, we can write $f = g + h$, where $g \in L^2$ and $\int_0^{2\pi} |h| < \epsilon$.

Proof. Since $f \in L^1$, there exists $\delta > 0$ such that for any subset $A \subseteq (0, 2\pi)$ with $|A| < \delta$, $\int_A |f| < \epsilon$. Take $M > 0$ sufficiently large such that

$$|\{|f| \geq M\}| < \delta.$$

Define

$$h(x) = \begin{cases} f(x) & \text{if } |f(x)| \geq M \\ 0 & \text{otherwise.} \end{cases}$$

Then $\int_0^{2\pi} |h| = \int_{\{|f| \geq M\}} |f| < \epsilon$. Clearly, $|g| \leq M$ and so $g \in L^2(0, 2\pi)$. □

Then $c_k(f) = c_k(g) + c_k(h)$, where $c_k(f) = \int_0^{2\pi} f\phi_k$. Note that $c_k(g) \to 0$ and

$$|c_k(h)| \leq \int_0^{2\pi} |h||\phi_k| \leq \frac{1}{\sqrt{\pi}} \int_0^{2\pi} |h| < \frac{\epsilon}{\sqrt{\pi}}$$

for all k.

Since $\epsilon > 0$ is arbitrary, thus $c_k(f) \to 0$ follows, that is,

$$\lim_{k\to\infty} \int_0^{2\pi} f(x) \cos kx \, dx = \lim_{k\to\infty} \int_0^{2\pi} f(x) \sin kx \, dx = 0.$$

8.6 Q16

Assume $f_k \to f$ in L^p norm, that is, $\|f_k - f\|_p \to 0$ as $k \to \infty$. For $g \in L^{p'}$,

$$\left| \int f_k g - \int fg \right| \leq \int |f_k - f||g| \leq \|f_k - f\|_p \|g\|_{p'}$$

by Hölder's inequality.

Since $\|g\|_{p'} < \infty$, thus $\int f_k g \to \int fg$.

Note by Exercise 15 that $\{\cos kx\}$ converges weakly in $L^2(0, 2\pi)$ to 0, since

$$\lim_{k\to\infty} \int_0^{2\pi} (\cos kx) g(x) \, dx = \int_0^{2\pi} (0) g(x) \, dx = 0$$

for $g \in L^2(0, 2\pi)$.

However for $k \in \mathbb{N}$,

$$\|\cos kx - 0\|_2^2 = \int_0^{2\pi} |\cos kx|^2 \, dx = \pi \not\to 0.$$

Hence, $\cos kx \not\to 0$ in L^2 norm.

8.7 Q17

$$\begin{aligned}
\|f_k - f\|_2^2 &= \int |f_k - f|^2 \\
&= \int (f_k^2 - 2f_k f + f^2) \\
&= \int f_k^2 - 2\int f_k f + \int f^2 \\
&= \|f_k\|_2^2 - 2\int f_k f + \|f\|_2^2.
\end{aligned}$$

Note that $\int f_k f \to \int f^2 = \|f\|_2^2$ and $\|f_k\|_2^2 \to \|f\|_2^2$.

Thus

$$\|f_k - f\|_2^2 \to \|f\|_2^2 - 2\|f\|_2^2 + \|f\|_2^2 = 0.$$

Hence $f_k \to f$ in L^2 norm.

8.8 Q21

Lemma 8.8.1. For $a, b \in \mathbb{R}$, $|a+b|^p \leq 2^p(|a|^p + |b|^p)$, where $0 < p < \infty$.

Proof.

$$\begin{aligned}
|a+b|^p &\leq (|a| + |b|)^p \\
&\leq (2\max\{|a|, |b|\})^p \\
&= 2^p(\max\{|a|, |b|\})^p \\
&\leq 2^p(|a|^p + |b|^p).
\end{aligned}$$

□

Let $\{r_k\}$ be the rational numbers. First note that for any Q, x, and r_k,

$$\frac{1}{|Q|}\int_Q |f(y)-f(x)|^p\,dy \le 2^p\frac{1}{|Q|}\int_Q |f(y)-r_k|^p\,dy + 2^p\frac{1}{|Q|}\int_Q |r_k - f(x)|^p\,dy$$
$$= 2^p\frac{1}{|Q|}\int_Q |f(y)-r_k|^p\,dy + 2^p|r_k - f(x)|^p.$$

Let Z_k be the set in which the formula

$$\lim_{Q\searrow x}\frac{1}{|Q|}\int_Q |f(y)-r_k|^p\,dy = |f(x)-r_k|^p$$

is *not* valid. Since

$$|f(y)-r_k|^p \le 2^p(|f(y)|^p + |r_k|^p)$$

is locally integrable, by Lebesgue's Differentiation Theorem, $|Z_k| = 0$. Let $Z = \bigcup Z_k$, then $|Z| = 0$.

Thus, if $x \notin Z$,

$$\limsup_{Q\searrow x}\frac{1}{|Q|}\int_Q |f(y)-f(x)|^p\,dy \le 2^{p+1}|f(x)-r_k|^p$$

for every r_k. For an x at which $f(x)$ is finite (in particular, almost everywhere since $f \in L^p(\mathbb{R}^n)$), by the density of rationals in $\mathbb{R}^n$ we can choose r_k such that $|f(x)-r_k|^p$ is arbitrarily small.

Thus

$$\limsup_{Q\searrow x}\frac{1}{|Q|}\int_Q |f(y)-f(x)|^p\,dy = 0 \quad \text{a.e.}$$

and this completes the proof, since $0 \le \liminf_{Q\searrow x}\frac{1}{|Q|}\int_Q |f(y)-f(x)|^p\,dy$ is clear.

(Note by Exercise 5 that if this condition holds for a given p, then it also holds for all smaller p.)

In Exercise 5, it is proved that if $p_1 < p_2$, then $N_{p_1}[f] \le N_{p_2}[f]$, where $N_p[f] = \left(\frac{1}{|E|}\int_E |f|^p\right)^{\frac{1}{p}}$. The proof is using Hölder's inequality to show

$$\int_E |f|^{p_1} \le \left(\int_E 1^{\frac{p_2}{p_2-p_1}}\right)^{\frac{p_2-p_1}{p_2}}\left(\int_E |f|^{p_1\cdot\frac{p_2}{p_1}}\right)^{\frac{p_1}{p_2}} = |E|^{1-\frac{p_1}{p_2}}\left(\int_E |f|^{p_2}\right)^{\frac{p_1}{p_2}}.$$

Thus, if the condition holds for a given p, for smaller $p_1 < p$,

$$\limsup_{Q \searrow x} \left(\frac{1}{|Q|} \int_Q |f(y) - f(x)|^{p_1} \, dy \right)^{\frac{1}{p_1}}$$
$$\leq \limsup_{Q \searrow x} \left(\frac{1}{|Q|} \int_Q |f(y) - f(x)|^{p} \, dy \right)^{\frac{1}{p}} = 0 \quad \text{a.e.}$$