Analysis Part 5

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Book: Measure and Integral by Wheeden and Zygmund

6 Chapter 6

6.1 Q6

By definition,

$$\widehat{(f*g)}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}(f*g)(t)e^{-ixt}\,dt.$$

Note that since $f, g \in L(\mathbb{R})$, Theorem 6.14 says that

$$\int_{-\infty}^{\infty}|(f*g)(t)|\,dt \leq \left(\int_{-\infty}^{\infty}|f|\,dx\right)\left(\int_{-\infty}^{\infty}|g|\,dx\right) < \infty.$$

Hence

$$\left|\int_{-\infty}^{\infty}(f*g)(t)e^{-ixt}\,dt\right| \leq \int_{-\infty}^{\infty}\left|(f*g)(t)e^{-ixt}\right|\,dt = \int_{-\infty}^{\infty}\left|(f*g)(t)\right|\,dt < \infty.$$

(For a complex-valued function $F = F_0 + iF_1$, note that if $\int |F| < \infty$, then $\int |F_0| \leq \int \sqrt{F_0^2 + F_1^2} = \int |F| < \infty$. Similarly $\int |F_1| < \infty$.)

Thus we may apply Fubini's Theorem to the real and imaginary parts of the function.

$$\int_{-\infty}^{\infty}(f*g)(t)e^{-ixt}\,dt = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(t-u)g(u)\,du\right)e^{-ixt}\,dt$$
$$= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(t-u)e^{-ixt}\,dt\right)g(u)\,du$$

(Fubini's Theorem)

$$= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(t-u)e^{-ix(t-u)}\,dt\right)g(u)e^{-ixu}\,du$$
$$= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(v)e^{-ixv}\,dv\right)g(u)e^{-ixu}\,du$$

(change of variables $v = t - u$)

$$= (2\pi\widehat{f}(x))\int_{-\infty}^{\infty} g(u)e^{-ixu}\,du$$
$$= (2\pi\widehat{f}(x))(2\pi\widehat{g}(x)).$$

Therefore

$$\widehat{(f*g)}(x) = 2\pi\widehat{f}(x)\widehat{g}(x).$$

6.2 Q10

We prove the statement by induction on n. First note that $v_1 = 2$, the length of the interval $[-1,1]$. Also, $v_2 = \pi$, the area of the unit circle. On the other hand,

$$2v_1\int_0^1 (1-t^2)^{\frac{1}{2}}\,dt = 4(\frac{\pi}{4}) = \pi.$$

Thus the formula is true for $n = 2$.

Suppose the formula is true for $n-1$. Let

$$B^n := \{\mathbf{x}\in\mathbb{R}^n : x_1^2+\cdots+x_n^2 \leq 1\}$$

denote the unit ball in $\mathbb{R}^n$.

$$
\begin{aligned}
v_n &= \int \cdots \int_{B^n} 1 \\
&= \int \cdots \int_{\{x_1^2+\cdots+x_n^2 \le 1\}} 1 \, dx_1 \ldots dx_n \\
&= \int_{-1}^{1} \left(\int \cdots \int_{\{x_2^2+\cdots+x_n^2 \le 1-x_1^2\}} 1 \, dx_2 \ldots dx_n \right) dx_1 \quad \text{(By Tonelli's Theorem).}
\end{aligned}
$$

Let $u_j = x_j/\sqrt{1-x_1^2}$ for $j = 2, \ldots, n$. Note that $\frac{du_j}{dx_j} = \frac{1}{\sqrt{1-x_1^2}}$. We make a change of variables:

$$
\begin{aligned}
&\int_{-1}^{1} \left(\int \cdots \int_{\{u_2^2+\cdots+u_n^2 \le 1\}} (1-x_1^2)^{(n-1)/2} \, du_2 \ldots du_n \right) dx_1 \\
&= \int_{-1}^{1} \left(\int \cdots \int_{\{u_2^2+\cdots+u_n^2 \le 1\}} du_2 \ldots du_n \right) (1-x_1^2)^{(n-1)/2} \, dx_1 \\
&= \int_{-1}^{1} (v_{n-1})(1-x_1^2)^{(n-1)/2} \, dx_1 \\
&= v_{n-1} \int_{-1}^{1} (1-x_1^2)^{(n-1)/2} \, dx_1 \\
&= 2v_{n-1} \int_{0}^{1} (1-x_1^2)^{(n-1)/2} \, dx_1
\end{aligned}
$$

(since the integrand is an even function)

$$
= 2v_{n-1} \int_0^1 (1-t^2)^{(n-1)/2} \, dt.
$$

By setting $w = t^2$, the integral becomes

$$
\begin{aligned}
\int_0^1 (1-t^2)^{(n-1)/2} \, dt &= \int_0^1 (1-w)^{(n-1)/2} (\frac{1}{2} w^{-1/2}) \, dw \\
&= \frac{1}{2} \int_0^1 w^{-1/2} (1-w)^{(n-1)/2} \, dw \\
&= \frac{1}{2} \int_0^1 w^{\frac{1}{2}-1} (1-w)^{(\frac{n+1}{2})-1} \, dw \\
&= \frac{1}{2} B(\frac{1}{2}, \frac{n+1}{2})
\end{aligned}
$$

where $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$ is the β-function, and thus can be expressed in terms of the Γ-function since $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$.

7 Chapter 7

7.1 Q1

Let $f \neq 0$ on E, where $|E| > 0$. Let $Q_r(x)$ denote the cube centered at x, with edge length r.

Let $E_k = E \cap Q_k(0)$, which is measurable. Since $E_k \nearrow E$, by MCT for measure, $\lim_{k\to\infty} |E_k| = |E| > 0$. In particular there exists $K > 0$ such that $|E_K| > 0$.

Thus

$$b := \int_{Q_K(0)} |f(y)|\,dy \geq \int_{E_K} |f(y)|\,dy > 0$$

since $|f| > 0$ on $E_K \subseteq E$.

Then

$$\begin{aligned} f^*(x) &= \sup \frac{1}{|Q|} \int_Q |f(y)|\,dy \\ &\geq \frac{1}{|Q_{K+2\|x\|_\infty}(x)|} \int_{Q_{K+2\|x\|_\infty}(x)} |f(y)|\,dy \\ &\geq \frac{1}{(K+2\|x\|_\infty)^n} \int_{Q_K(0)} |f(y)|\,dy \end{aligned}$$

where $\|x\|_\infty := \max(|x_1|, \ldots, |x_n|)$.

The last inequality follows since $Q_K(0) \subseteq Q_{K+2\|x\|_\infty}(x)$.

Since $\|x\|_\infty \le |x|$ for all $x \in \mathbb{R}^n$, thus for $|x| \ge 1$,

$$\begin{aligned} f^*(x) &\ge \frac{b}{(K+2|x|)^n} \\ &\ge \frac{b}{(K|x|+2|x|)^n} && \text{(since } |x| \ge 1) \\ &= \frac{b}{(K+2)^n|x|^n}. \end{aligned}$$

7.2 Q2

Lemma 7.2.1. We show that $\int \phi_\epsilon = 1$.

Proof. For $\epsilon > 0$, note that

$$\int_{\mathbb{R}^n} \phi_\epsilon(x)\,dx = \int_{\mathbb{R}^n} \epsilon^{-n}\phi(x/\epsilon)\,dx = \int_{\{|x|<\epsilon\}} \epsilon^{-n}\phi(x/\epsilon)\,dx$$

since $\phi(x) = 0$ for $|x| \ge 1$.

Let $y = Tx = \frac{1}{\epsilon}x$ be a linear transformation of $\mathbb{R}^n$. Note that $T = \operatorname{diag}(\frac{1}{\epsilon}, \ldots, \frac{1}{\epsilon})$ so that $|\det T| = \epsilon^{-n}$. If $E = \{x \in \mathbb{R}^n : |x| < 1\}$, note that $T^{-1}E = \{x \in \mathbb{R}^n : |x| < \epsilon\}$.

Thus using the formula

$$\int_E f(y)\,dy = |\det T| \int_{T^{-1}E} f(Tx)\,dx$$

proved in Chapter 5 Exercise 20, we get

$$\begin{aligned} \int_{\mathbb{R}^n} \phi_\epsilon(x)\,dx &= \epsilon^{-n} \int_{T^{-1}E} \phi(Tx)\,dx \\ &= \epsilon^{-n} \cdot \frac{1}{|\det T|} \int_E \phi(y)\,dy \\ &= \int_{\{|y|<1\}} \phi(y)\,dy \\ &= \int_{\mathbb{R}^n} \phi(y)\,dy \\ &= 1. \end{aligned}$$

Then,

$$
\begin{aligned}
(f * \phi_\epsilon)(x) - f(x) &= \int_{\mathbb{R}^n} f(x-y)\phi_\epsilon(y)\,dy - \int_{\mathbb{R}^n} f(x)\phi_\epsilon(y)\,dy \\
&= \int_{\mathbb{R}^n} [f(x-y) - f(x)]\phi_\epsilon(y)\,dy \\
&= \frac{1}{\epsilon^n}\int_{\{|y|\le\epsilon\}} [f(x-y) - f(x)]\phi(y/\epsilon)\,dy.
\end{aligned}
$$

Since $|\phi(x)| \le M$ for some $M > 0$, we have that

$$
\begin{aligned}
|(f * \phi_\epsilon)(x) - f(x)| &\le \frac{M}{\epsilon^n}\int_{\{|y|\le\epsilon\}} |f(x-y) - f(x)|\,dy \\
&= \frac{M}{\epsilon^n}\int_{\{|y-x|\le\epsilon\}} |f(y) - f(x)|\,dy \\
&\le \frac{M}{\epsilon^n}\int_{Q_{2\epsilon}(x)} |f(y) - f(x)|\,dy \\
&\text{(where } Q_{2\epsilon}(x) \text{ is the cube centered at } x \text{ with edge length } 2\epsilon) \\
&\le \frac{2^n M}{|Q_{2\epsilon}(x)|}\int_{Q_{2\epsilon}(x)} |f(y) - f(x)|\,dy \\
&\text{(since } |Q_{2\epsilon}(x)| = 2^n\epsilon^n).
\end{aligned}
$$

We quote Theorem 7.16:

Theorem (Theorem 7.16)**.** Let f be locally integrable in $\mathbb{R}^n$. Then at every point x of the Lebesgue set of f (in particular, almost everywhere), $\frac{1}{|S|}\int_S |f(y) - f(x)|\,dy \to 0$ for any family $\{S\}$ that shrinks regularly to x. Thus, also $\frac{1}{|S|}\int_S f(y)\,dy \to f(x)$ a.e.

Since $f \in L(\mathbb{R}^n)$, by Theorem 7.16, at every point x of the Lebesgue set of f,

$$
\frac{1}{|Q_{2\epsilon}(x)|}\int_{Q_{2\epsilon}(x)} |f(y) - f(x)|\,dy \to 0
$$

as $\epsilon \to 0$.

Hence $\lim_{\epsilon\to 0}|(f*\phi_\epsilon)(x)-f(x)|=0$, which implies

$$\lim_{\epsilon\to 0}(f*\phi_\epsilon)(x)=f(x)$$

in the Lebesgue set of f. □

7.3 Q5

Lemma 7.3.1. $\int_a^b \phi\,df=\int_a^b \phi\,dg+\int_a^b \phi\,dh$.

Proof. Firstly, note that g is absolutely continuous implies g is of bounded variation on $[a,b]$. Thus $h=f-g$ is also of bounded variation on $[a,b]$. Thus the above three integrals are well-defined.

Then

$$\begin{aligned}\int_a^b \phi\,df &= \lim_{P\to 0}\sum \phi(\xi_i)(f(x_i)-f(x_{i-1}))\\ &= \lim_{P\to 0}\sum \phi(\xi_i)(g(x_i)+h(x_i)-g(x_{i-1})-h(x_{i-1}))\\ &= \lim_{P\to 0}\sum \phi(\xi_i)(g(x_i)-g(x_{i-1}))+\lim_{P\to 0}\sum \phi(\xi_i)(h(x_i)-h(x_{i-1}))\\ &= \int_a^b \phi\,dg+\int_a^b \phi\,dh.\end{aligned}$$

□

We quote Theorem 7.32:

Theorem (Theorem 7.32(i))**.** If g is continuous on $[a,b]$ and f is absolutely continuous on $[a,b]$, then $\int_a^b g\,df=\int_a^b gf'\,dx$.

Applying Theorem 7.32, we get

$$\int_a^b \phi\,dg=\int_a^b \phi g'\,dx=\int_a^b \phi f'\,dx$$

since $f'=g'+h'=g'$ a.e. on $[a,b]$.

Combining our results, we have

$$\int_a^b \phi\,df=\int_a^b \phi f'\,dx+\int_a^b \phi\,dh.$$

7.4 Q8

Since f is of bounded variation on $[a, b]$,

$$V(x) = V[a, x] \le V[a, b] < \infty$$

for all $x \in [a, b]$.

Since $V(x)$ is absolutely continuous on $[a, b]$, for given $\epsilon > 0$, there exists $\delta > 0$ such that for any collection $\{[a_i, b_i]\}$ of nonoverlapping subintervals of $[a, b]$,

$$\sum |V(b_i) - V(a_i)| < \epsilon \quad \text{if} \quad \sum (b_i - a_i) < \delta.$$

Note that $V(b_i) - V(a_i)$ is well-defined since $V(x) < \infty$ for all $x \in [a, b]$.

Then, if $\sum (b_i - a_i) < \delta$,

$$\begin{aligned}\sum |f(b_i) - f(a_i)| &\le \sum V[a_i, b_i] \\ &= \sum |V(b_i) - V(a_i)| \\ &\text{(since } V(b_i) - V(a_i) = V[a, b_i] - V[a, a_i] = V[a_i, b_i]) \\ &< \epsilon.\end{aligned}$$

Thus f is absolutely continuous on $[a, b]$.

7.5 Q10

(a)

Lemma 7.5.1. The image of an interval $[a_i, b_i] \subseteq [a, b]$ under f is an interval of length at most $V(b_i) - V(a_i) = V[a_i, b_i]$.

Proof. Since f is continuous on $[a_i, b_i]$, $f([a_i, b_i])$ is an interval by the Intermediate Value Theorem. By the Extreme Value Theorem, $f([a_i, b_i]) = [f(p), f(q)]$ for some $p, q \in [a_i, b_i]$. Then

$$|f([a_i, b_i])| = f(q) - f(p) \le V[a_i, b_i] = V(b_i) - V(a_i).$$

□

Lemma 7.5.2. Let $Z \subseteq [a, b]$ be of measure zero. Then $f(Z)$ also has measure zero.

Proof. Let $\epsilon > 0$. Since f is absolutely continuous, $V(x)$ is absolutely continuous on $[a, b]$. Choose δ such that $\sum |V(b_i) - V(a_i)| < \epsilon$ for any collection of nonoverlapping intervals with $\sum(b_i - a_i) < \delta$.

Let G be an open set containing Z such that $|G| = |G \setminus Z| < \delta$. Write $G = \bigcup_{i=1}^{\infty} J_i$, where J_i are nonoverlapping closed intervals. Let $[a_i, b_i] = J_i \cap [a, b]$, then $\{[a_i, b_i]\}$ are nonoverlapping subintervals of $[a, b]$ with $\sum(b_i - a_i) \leq |G| < \delta$.

Note that $Z \subseteq \bigcup_{i=1}^{\infty}[a_i, b_i]$ so that $f(Z) \subseteq \bigcup_{i=1}^{\infty} f([a_i, b_i])$.

Note that

$$\begin{aligned} |\bigcup_{i=1}^{\infty} f([a_i, b_i])| &\leq \sum_{i=1}^{\infty} |f([a_i, b_i])| \\ &\leq \sum_{i=1}^{\infty} |V(b_i) - V(a_i)| \\ &< \epsilon. \end{aligned}$$

Thus $|f(Z)|_e \leq |\bigcup_{i=1}^{\infty} f([a_i, b_i])| < \epsilon$. Since $\epsilon > 0$ is arbitrary, $f(Z)$ has measure zero. □

Let E be any measurable subset of $[a, b]$. Write $E = H \cup Z$, where H is an F_σ set and $|Z| = 0$. Write $H = \bigcup_{i=1}^{\infty} F_i$ where $F_i \subseteq [a, b]$ are closed (and bounded thus compact) sets. Then

$$f(H) = \bigcup_{i=1}^{\infty} f(F_i)$$

is measurable since each $f(F_i)$ is compact and thus measurable.

Hence $f(E) = f(H) \cup f(Z)$ is measurable.

(b)

(In latest 2015 edition of the book) Give an example of a strictly increasing Lipschitz continuous function f and a set Z with measure 0 such that $f^{-1}(Z)$ does not have measure 0 (and consequently, f^{-1} is not absolutely continuous). (Let $f^{-1}(x) = x + C(x)$ on $[0, 1]$, where $C(x)$ is the Cantor-Lebesgue function.)

Solution. Let $f^{-1}(x) = x+C(x)$ on $[0, 1]$. Since $f^{-1}(x)$ is strictly increasing, its inverse $f(x)$ exists and f is strictly increasing.

Let $x, y \in f^{-1}([0, 1]) = [0, 2]$. Suppose $x < y$ and write $x = p + C(p)$, $y = q + C(q)$ where $p < q$. Then

$$\begin{aligned} f(y) - f(x) &= f(q + C(q)) - f(p + C(p)) \\ &= q - p \\ &\text{(since } f^{-1}(x) = x + C(x) \text{ implies } x = f(x + C(x))) \\ &\leq q + C(q) - p - C(p) \\ &\text{(since the Cantor function is increasing)} \\ &= y - x. \end{aligned}$$

So f is Lipschitz continuous.

Let Z be the Cantor set, which has measure zero.

Lemma 7.5.3. $f^{-1}(Z)$ does not have measure 0.

Proof. Since $C(x)$ is constant on each disjoint interval in $[0, 1] \setminus Z$, $f^{-1}(x)$ maps each interval to an interval of the same length. Thus $|f^{-1}([0, 1] \setminus Z)| = |[0, 1] \setminus Z| = 1$. Since $f^{-1}([0, 1]) = [0, 2]$, thus $|f^{-1}(Z)| = 1 > 0$. □

7.6 Q11

Lemma 7.6.1. When f is the characteristic function of an interval, the statement is true.

Proof. Let $f = \chi_{[c,d]}$, where $[c,d] \subseteq [a,b]$. Let $c = g(\gamma)$, $d = g(\delta)$. We have that $f(g(t))g'(t) = g'(t)$ for $t \in [\gamma, \delta]$, and 0 elsewhere since monotone continuous g maps $[\gamma, \delta]$ to $[c,d]$.

Since g is monotone increasing and absolutely continuous on $[\alpha, \beta]$, $g' \geq 0$ exists a.e. in $[\alpha, \beta]$, and g' is integrable on $[\alpha, \beta]$. Hence $f(g(t))g'(t)$ is measurable on $[\alpha, \beta]$.

$$\begin{aligned}\int_a^b f(x)\,dx &= d - c \\ &= g(\delta) - g(\gamma) \\ &= \int_\gamma^\delta \chi_{[c,d]}(g(t))g'(t)\,dt \\ &= \int_\alpha^\beta f(g(t))g'(t)\,dt.\end{aligned}$$

□

Lemma 7.6.2. When $f = \chi_G$, where $G \subseteq [a,b]$ is open, the statement is true.

Proof. Write $G = \bigcup_{i=1}^{\infty} J_i$, where J_i are nonoverlapping closed intervals. Then $f = \chi_G = \sum \chi_{J_i}$. Also,

$$f(g(t))g'(t) = \sum \chi_{J_i}(g(t))g'(t)$$

which is the (limit of) sum of measurable functions thus measurable. Thus from Lemma 7.6.1,

$$\begin{aligned}\int_a^b f(x)\,dx &= \sum \int_a^b \chi_{J_i}\,dx \\ &= \sum \int_\alpha^\beta \chi_{J_i}(g(t))g'(t)\,dt \\ &= \int_\alpha^\beta f(g(t))g'(t)\,dt.\end{aligned}$$

□

Lemma 7.6.3. If $f = \chi_G$, where G is a G_δ set, the statement is true.

Proof. Write $G = \bigcap_{i=1}^{\infty} G_i \subseteq [a,b]$, where G_i are open sets. We may choose G_i such that $G_i \searrow G$. Thus $\chi_{G_i} \searrow \chi_G$. Since $g'(t) \geq 0$ a.e., we also have that $\chi_{G_i}(g(t))g'(t) \searrow \chi_G(g(t))g'(t)$ a.e.

Since by Lemma 7.6.2 each $\chi_{G_i}(g(t))g'(t)$ is measurable, $\chi_G(g(t))g'(t)$ is measurable. Note that $\chi_{G_i}(g(t))g'(t) \leq g'(t)$ a.e. and $g'(t)$ is integrable.

By Monotone Convergence Theorem, $\int_a^b \chi_{G_i} \to \int_a^b \chi_G$ and $\int_\alpha^\beta \chi_{G_i}(g(t))g'(t) \to \int_\alpha^\beta \chi_G(g(t))g'(t)$.

Hence

$$\begin{aligned}
\int_a^b f(x)\,dx &= \lim_{i\to\infty} \int_a^b \chi_{G_i}(x)\,dx \\
&= \lim_{i\to\infty} \int_\alpha^\beta \chi_{G_i}(g(t))g'(t)\,dt \qquad \text{(by Lemma 7.6.2)} \\
&= \int_\alpha^\beta \chi_G(g(t))g'(t)\,dt \\
&= \int_\alpha^\beta f(g(t))g'(t)\,dt.
\end{aligned}$$

□

Lemma 7.6.4. If $f = \chi_Z$ where $|Z| = 0$, the statement is true.

Proof. Clearly $\int_a^b f(x)\,dx = 0$. Since Z is measurable, there is a G_δ-set $G \supseteq Z$ with $|G| = 0$. Then

$$0 \leq \chi_Z(g(t))g'(t) \leq \chi_G(g(t))g'(t) = 0 \text{ a.e.}$$

since $\int_\alpha^\beta \chi_G(g(t))g'(t)\,dt = \int_a^b \chi_G(x)\,dx = 0$ by Lemma 7.6.3.

Hence $\chi_Z(g(t))g'(t) = 0$ a.e. (and thus measurable) which implies

$$\int_\alpha^\beta f(g(t))g'(t)\,dt = 0 = \int_a^b f(x)\,dx.$$

□

Lemma 7.6.5. If $f = \chi_E$, where E is a measurable subset of $[a, b]$, the statement is true.

Proof. Write $E = G \backslash Z$ where G is a G_δ-set and $|Z| = 0$. Hence $\chi_E(g(t))g'(t) = \chi_G(g(t))g'(t) - \chi_Z(g(t))g'(t)$ is measurable. Thus from Lemma 7.6.3 and Lemma 7.6.4, we conclude that

$$\int_a^b f(x)\,dx = \int_a^b \chi_G(x)\,dx = \int_\alpha^\beta \chi_G(g(t))g'(t)\,dt = \int_\alpha^\beta f(g(t))g'(t)\,dt.$$

□

Now, if $f = \sum_{i=1}^n a_i \chi_{E_i}$ is a simple function, the statement is true by linearity of the integral. Note that $f(g(t))g'(t)$ is the sum of measurable functions thus measurable.

If $f \geq 0$, f can be written as a limit of increasing non-negative simple functions $f_k \nearrow f$. Note that $f(g(t))g'(t)$ is the limit of measurable functions, thus measurable.

Then

$$\begin{aligned}\int_a^b f(x)\,dx &= \lim_{k\to\infty}\int_a^b f_k(x)\,dx \quad \text{(by MCT)}\\ &= \lim_{k\to\infty}\int_\alpha^\beta f_k(g(t))g'(t)\,dt\\ &= \int_\alpha^\beta f(g(t))g'(t)\,dt\end{aligned}$$

(by MCT since $f_k(g(t))g'(t)$ are nonnegative and increase to $f(g(t))g'(t)$.)

Finally, for arbitrary integrable f on $[a, b]$, write $f = f^+ - f^-$, where $f^+ \geq 0$ and $f^- \geq 0$. The result then follows from the previous paragraph. Done.

7.7 Q16

Let f be the Cantor-Lebesgue function on $[0,1]$, and let $f=0$ elsewhere.

Since f is monotone on $[0,1]$, $V[f;0,1]=f(1)-f(0)=1<\infty$. Since f clearly has zero variation on any interval outside $[0,1]$, thus f is of bounded variation on $(-\infty,\infty)$.

Define the bump function $g:\mathbb{R}\to\mathbb{R}$ by

$$g(x)=\begin{cases}\exp(-\frac{1}{1-x^2}) & \text{for } |x|<1\\ 0 & \text{otherwise.}\end{cases}$$

It is clear that g has compact support. It can be shown that g is infinitely differentiable, and $g^{(n)}(x)=0$ for $x=\pm 1$.

For $|x|<1$,

$$g'(x)=\frac{e^{-\frac{1}{1-x^2}}(-2x)}{(1-x^2)^2}.$$

So, for $0<x<1$, $g'(x)<0$.

Hence

$$\begin{aligned}\int_{-\infty}^{\infty} fg' &\leq \int_{1/3}^{2/3} fg' && (\text{since } fg'\leq 0 \text{ on } \mathbb{R})\\ &= \int_{1/3}^{2/3} \frac{1}{2}g' && (\text{since } f=\frac{1}{2} \text{ on } [\frac{1}{3},\frac{2}{3}])\\ &< 0.\end{aligned}$$

However,

$$-\int_{-\infty}^{\infty} f'g = 0$$

since $f'=0$ a.e.

7.8 Q17

Let $\{f_k\}$ be a sequence of integrable functions on $(0,1)$ which converges pointwise a.e. to an integrable f.

($\implies$) Assume $\int_0^1 |f - f_k| \to 0$.

Lemma 7.8.1. Given $\epsilon > 0$, there exists $\delta > 0$ such that if $E \subseteq (0,1)$ satisfies $|E| < \delta$, then $|\int_E f| \leq \int_E |f| < \epsilon$.

Proof. Define $A_k = \{x \in (0,1) : \frac{1}{k} \leq |f(x)| < k\}$ for $k \in \mathbb{N}$. Each A_k is measurable and $A_k \nearrow A := \bigcup_{k=1}^{\infty} A_k$. Note that

$$\int_0^1 |f| = \int_{\{f=0\}} |f| + \int_A |f| + \int_{\{f=\infty\}} |f| = \int_A |f|$$

since $\int_{\{f=0\}} |f| = 0$ and $\int_{\{f=\infty\}} |f| = 0$ since f is integrable.

Let $g_k = |f|\chi_{A_k}$. Then $\{g_k\}$ is a sequence of non-negative functions such that $g_k \nearrow |f|\chi_A$. By Monotone Convergence Theorem, $\lim_{k\to\infty} \int_0^1 g_k = \int_0^1 |f|\chi_A$, that is,

$$\lim_{k\to\infty} \int_{A_k} |f|\, dx = \int_A |f|\, dx = \int_0^1 |f|\, dx.$$

Let $N > 0$ be sufficiently large such that $\int_{(0,1)\setminus A_N} |f|\, dx < \epsilon/2$. Let $\delta = \frac{\epsilon}{2N}$, and suppose $|E| < \delta$. Then

$$\begin{aligned}
\left|\int_E f\, dx\right| &\leq \int_E |f|\, dx \\
&= \int_{((0,1)\setminus A_N)\cap E} |f|\, dx + \int_{A_N\cap E} |f|\, dx \\
&\leq \int_{(0,1)\setminus A_N} |f|\, dx + \int_{A_N\cap E} N\, dx \\
&< \frac{\epsilon}{2} + N \cdot |A_N \cap E| \\
&\leq \frac{\epsilon}{2} + N \cdot |E| \\
&< \frac{\epsilon}{2} + N \cdot \frac{\epsilon}{2N} \\
&= \epsilon.
\end{aligned}$$

□

Similarly, there exists $\delta_k > 0$ such that for $|E| < \delta_k$, $|\int_E f_k| < \epsilon$, for each k.

Since $\int_0^1 |f - f_k| \to 0$, there exists $N > 0$ such that $\int_0^1 |f - f_k| < \epsilon$ for all $k > N$.

Take $\delta' = \min\{\delta_1, \ldots, \delta_N, \delta\}$. Let $|E| < \delta'$.

For $1 \le k \le N$, clearly $|\int_E f_k| < \epsilon$ since $|E| < \delta' \le \delta_k$.

For $k > N$,

$$\begin{aligned}\left|\int_E f_k\right| &\le \int_E |f_k| \\ &\le \int_E |f_k - f| + \int_E |f| \\ &\le \int_0^1 |f - f_k| + \int_E |f| \\ &< \epsilon + \epsilon.\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, this shows that the indefinite integrals of the f_k are uniformly absolutely continuous.

($\Longleftarrow$) Assume the indefinite integrals of the f_k are uniformly absolutely continuous. Then given $\epsilon > 0$, there exists $\delta > 0$ such that if $|E| < \delta$, then $|\int_E f_k| < \epsilon$ for all k.

Lemma 7.8.2. We can strengthen the condition to: given $\epsilon > 0$, there exists $\delta > 0$ such that if $|E| < \delta$, then $\int_E |f_k| < \epsilon$ for all k.

Proof. Let $\epsilon > 0$. Choose $\delta > 0$ such that if $|E| < \delta$, then $|\int_E f_k| < \epsilon/2$ for all k. Then,

$$\begin{aligned}\int_E |f_k| &= \int_{E \cap \{f_k \ge 0\}} f_k - \int_{E \cap \{f_k < 0\}} f_k \\ &\le \left|\int_E f_k\right| + \left|\int_E f_k\right| \\ &< \epsilon.\end{aligned}$$

□

Recall that we showed there exists $\delta' > 0$ such that if $|E| < \delta'$, then $\int_E |f| < \epsilon$.

Since $f_k \to f$ a.e. on $(0,1)$ and $|(0,1)| < \infty$, thus $f_k \xrightarrow{m} f$ on $(0,1)$. That is, there exists $N > 0$ such that for $k > N$,

$$|\{x \in (0,1) : |f(x) - f_k(x)| > \epsilon\}| < \min\{\delta, \delta'\}.$$

Let $E_k := \{x \in (0,1) : |f(x) - f_k(x)| > \epsilon\}$. Then for $k > N$,

$$\begin{aligned}\int_0^1 |f - f_k| &= \int_{E_k} |f - f_k| + \int_{(0,1)\setminus E_k} |f - f_k| \\ &\leq \int_{E_k} |f| + \int_{E_k} |f_k| + \int_{(0,1)\setminus E_k} \epsilon \\ &< \epsilon + \epsilon + \epsilon \\ &= 3\epsilon.\end{aligned}$$

Thus $\int_0^1 |f - f_k| \to 0$ as desired.