Analysis Part 4

www.mathtuition88.com

Book: Measure and Integral by Wheeden and Zygmund

5 Chapter 5

5.1 Q10

Assume $p > 0$, $\int_E |f - f_k|^p \to 0$, and $\int_E |f_k|^p \leq M$ for all k.

By Q9, there is a subsequence $f_{k_j} \to f$ a.e. in E. Since $t \mapsto |t|^p$ is continuous, $|f_{k_j}|^p \to |f|^p$ a.e. in E.

By Fatou's Lemma,

$$\int_E |f|^p \leq \liminf_{j\to\infty} \int_E |f_{k_j}|^p \leq M.$$

5.2 Q14

Let $f \in L^p(E)$. WLOG we may assume $f \geq 0$ since for $a > 0$,

$$0 \leq a^p \omega(a) = a^p |\{f > a\}| \leq a^p |\{|f| > a\}|.$$

Thus if $\lim_{a\to 0+} a^p |\{|f| > a\}| = 0$, that would imply $\lim_{a\to 0+} a^p \omega(a) = 0$.

Let $\epsilon > 0$.

Lemma 5.1. We may choose $\delta > 0$ such that $\int_{\{f\leq\delta\}} f^p < \epsilon$.

Proof. Define
$$f_k(x) = \begin{cases} 0 & \text{if } 0 \le f(x) \le \frac{1}{k} \\ f(x) & \text{otherwise.} \end{cases}$$
Then $f_k^p \to f^p$ and $|f_k^p| \le f^p$. Since $f^p \in L(E)$, by Lebesgue's DCT, $\int_E f_k^p \to \int_E f^p$. There exists K such that for $k \ge K$, $|\int_E f^p - \int_E f_k^p| = |\int_{\{f \le \frac{1}{k}\}} f^p| < \epsilon$. Take $\delta = \frac{1}{K}$, then $\int_{\{f \le \delta\}} f^p < \epsilon$. □

Note that $\omega(a), \omega(\delta) < \infty$ since $f \in L^p(E)$. Thus
$$\begin{aligned} a^p[\omega(a) - \omega(\delta)] &= a^p[|\{f > a\}| - |\{f > \delta\}|] \\ &= a^p|\{a < f \le \delta\}| \\ &= \int_{\{a<f\le\delta\}} a^p \\ &\le \int_{\{a<f\le\delta\}} f^p \\ &< \epsilon \end{aligned}$$
for $0 < a < \delta$.

Rearranging, we get $a^p\omega(a) < \epsilon + a^p\omega(\delta)$. Letting $a \to 0+$ gives
$$\lim_{a\to 0+} a^p\omega(a) = 0$$
since $\epsilon > 0$ is arbitrary.

5.3 Q15

Since $\omega(\alpha)$ is a decreasing function, for $a > 0$ we have
$$\begin{aligned} \int_{a/2}^{a} \alpha^{p-1}\omega(\alpha)\, d\alpha &\ge \omega(a) \int_{a/2}^{a} \alpha^{p-1}\, d\alpha \\ &= \omega(a)\left[\frac{\alpha^p}{p}\right]_{a/2}^{a} \\ &= \omega(a)a^p(\frac{2^p - 1}{2^p p}). \end{aligned}$$

Thus,

$$a^p\omega(a) \le \frac{2^p p}{2^p-1}\int_{a/2}^{a} \alpha^{p-1}\omega(\alpha)\,d\alpha \le \frac{2^p p}{2^p-1}\int_{0}^{a} \alpha^{p-1}\omega(\alpha)\,d\alpha.$$

Lemma 5.2. $\lim_{a\to 0+}\int_0^a \alpha^{p-1}\omega(\alpha)\,d\alpha = 0$.

Proof. For $0 < a < 1$, we have

$$\int_0^a \alpha^{p-1}\omega(\alpha)\,d\alpha = \int_0^1 \alpha^{p-1}\omega(\alpha)\,d\alpha - \int_a^1 \alpha^{p-1}\omega(\alpha)\,d\alpha.$$

Note that $\int_a^1 \alpha^{p-1}\omega(\alpha)\,d\alpha = \int_0^1 \alpha^{p-1}\omega(\alpha)\cdot\chi_{[a,1]}\,d\alpha$. Let $a_k \to 0+$, then note that

$$0 \le \alpha^{p-1}\omega(\alpha)\chi_{[a_k,1]} \nearrow \alpha^{p-1}\omega(\alpha)$$

on $(0,1)$ thus by Monotone Convergence Theorem,

$$\int_a^1 \alpha^{p-1}\omega(\alpha)\,d\alpha \to \int_0^1 \alpha^{p-1}\omega(\alpha)\,d\alpha$$

as $a \to 0+$. This proves $\lim_{a\to 0+}\int_0^a \alpha^{p-1}\omega(\alpha)\,d\alpha = 0$. □

Hence $\lim_{a\to 0+} a^p\omega(a) = 0$ as a direct consequence of the lemma.

Similarly for $b > 0$ we have

$$b^p\omega(b) \le \frac{2^p p}{2^p-1}\int_{b/2}^{b} \alpha^{p-1}\omega(\alpha)\,d\alpha.$$

Lemma 5.3. $\lim_{b\to\infty}\int_{b/2}^b \alpha^{p-1}\omega(\alpha)\,d\alpha = 0$.

Proof. Write

$$\int_{b/2}^b \alpha^{p-1}\omega(\alpha)\,d\alpha = \int_0^b \alpha^{p-1}\omega(\alpha)\,d\alpha - \int_0^{b/2} \alpha^{p-1}\omega(\alpha)\,d\alpha.$$

By similar argument using Monotone Convergence Theorem, we have

$$\lim_{b\to\infty}\int_0^b \alpha^{p-1}\omega(\alpha)\,d\alpha = \lim_{b\to\infty}\int_0^{b/2} \alpha^{p-1}\omega(\alpha)\,d\alpha = \int_0^\infty \alpha^{p-1}\omega(\alpha)\,d\alpha < \infty.$$

Thus $\lim_{b\to\infty}\int_{b/2}^b \alpha^{p-1}\omega(\alpha)\,d\alpha = 0$. □

This proves $\lim_{b\to\infty} b^p\omega(b) = 0$.

5.4 Q16

Let $E_{ab} = \{x \in E : a < f(x) \leq b\}$ for $0 < a < b < \infty$. We quote a theorem from the textbook:

Theorem (Theorem 5.46)**.** If $a < f \leq b$ (a and b finite) in E ($|E| < \infty$) and ϕ is continuous on $[a, b]$, then $\int_E \phi(f) = -\int_a^b \phi(\alpha)\, d\omega(\alpha)$.

Note that $|E_{ab}| \leq \omega(a) < \infty$ and $\phi(\alpha) = \alpha^p$ is continuous. Applying Theorem 5.46, we have

$$\int_{E_{ab}} f^p = -\int_a^b \alpha^p \, d\omega(\alpha).$$

Taking limits as $a \to 0+$ and $b \to \infty$, we get

$$\int_E f^p = -\int_0^\infty \alpha^p \, d\omega(\alpha)$$

by Monotone Convergence Theorem, since $f^p \chi_{E_{ab}} \nearrow f^p$ on E.

If $\int_0^\infty \alpha^p \, d\omega(\alpha) = -\infty$ and $\int_0^\infty \alpha^{p-1}\omega(\alpha)\, d\alpha = \infty$, then Theorem 5.51 holds since

$$\infty = \int_E f^p = -\int_0^\infty \alpha^p \, d\omega(\alpha) = p\int_0^\infty \alpha^{p-1}\omega(\alpha)\, d\alpha.$$

Next, assume either $\int_0^\infty \alpha^p \, d\omega(\alpha)$ or $\int_0^\infty \alpha^{p-1}\omega(\alpha)$ is finite.

By Theorem 2.21 (integration by parts), if $0 < a < b < \infty$, we have

$$-\int_a^b \alpha^p \, d\omega(\alpha) = -b^p\omega(b) + a^p\omega(a) + p\int_a^b \alpha^{p-1}\omega(\alpha)\, d\alpha \tag{5.1}$$

using the fact that α^p is continuously differentiable on $[a, b]$.

Case 1) If $\int_0^\infty \alpha^{p-1}\omega(\alpha)\, d\alpha < \infty$, using Q15 and taking limits as $a \to 0+$, $b \to \infty$ in (5.1), we get

$$-\int_0^\infty \alpha^p \, d\omega(\alpha) = 0 + 0 + p\int_0^\infty \alpha^{p-1}\omega(\alpha)\, d\alpha.$$

Case 2) If $|\int_0^\infty \alpha^p \, d\omega(\alpha)| < \infty$, then $\int_E f^p = -\int_0^\infty \alpha^p \, d\omega(\alpha) < \infty$, i.e. $f \in L^p(E)$. Thus Lemma 5.50 and Q14 holds so that $\lim_{b\to\infty} b^p\omega(b) = \lim_{a\to 0+} a^p\omega(a) = 0$. Hence taking limits as $a \to 0+$, $b \to \infty$ in (5.1), we get

$$-\int_0^\infty \alpha^p \, d\omega(\alpha) = p\int_0^\infty \alpha^{p-1}\omega(\alpha)\, d\alpha.$$

5.5 Q18

Let $f \geq 0$. By Question 16,

$$\int_E f^p = p\int_0^\infty \alpha^{p-1}\omega(\alpha)\, d\alpha,$$

thus $f \in L^p$ iff $\int_0^\infty \alpha^{p-1}\omega(\alpha)\, d\alpha < \infty$.

The key observation is

$$\int_0^\infty \alpha^{p-1}\omega(\alpha)\, d\alpha = \sum_{k=-\infty}^{\infty} \int_{2^k}^{2^{k+1}} \alpha^{p-1}\omega(\alpha)\, d\alpha. \tag{5.2}$$

Since α^{p-1} is increasing and $\omega(\alpha)$ is decreasing, we have

$$\int_{2^k}^{2^{k+1}} (2^k)^{p-1}\omega(2^{k+1}) \leq \int_{2^k}^{2^{k+1}} \alpha^{p-1}\omega(\alpha) \leq \int_{2^k}^{2^{k+1}} (2^{k+1})^{p-1}\omega(2^k).$$

Simplifying, we get

$$2^{-p}[2^{(k+1)p}\omega(2^{k+1})] \leq \int_{2^k}^{2^{k+1}} \alpha^{p-1}\omega(\alpha) \leq 2^{p-1}[2^{kp}\omega(2^k)].$$

Summing from $k = -\infty$ to $k = \infty$, we get

$$2^{-p}\sum_{k=-\infty}^{\infty} 2^{(k+1)p}\omega(2^{k+1}) \leq \int_0^\infty \alpha^{p-1}\omega(\alpha)\, d\alpha \leq 2^{p-1}\sum_{k=-\infty}^{\infty} 2^{kp}\omega(2^k).$$

Note that the left most term

$$2^{-p}\sum_{k=-\infty}^{\infty} 2^{(k+1)p}\omega(2^{k+1}) = 2^{-p}\sum_{k=-\infty}^{\infty} 2^{kp}\omega(2^k)$$

by change of index in summation.

Therefore $\sum_{k=-\infty}^{\infty} 2^{kp}\omega(2^k) < \infty$ iff $\int_0^\infty \alpha^{p-1}\omega(\alpha)\, d\alpha < \infty$ iff $f \in L^p$.

5.6 Q20

We first prove the statement for any indicator function χ_{E_1}, where $E_1 \subseteq E$ is measurable. We will use the following theorem.

Theorem (Theorem 3.35)**.** Let T be a linear transformation of $\mathbb{R}^n$, and let E be measurable. Then $|TE| = |\det T||E|$.

We have

$$
\begin{aligned}
|\det T| \int_{T^{-1}E} \chi_{E_1}(Tx)\,dx &= |\det T| \int_{T^{-1}E} \chi_{T^{-1}E_1}(x)\,dx \\
&= |\det T||T^{-1}E_1| \\
&= |\det T||\det T^{-1}||E_1| && \text{(by Theorem 3.35)} \\
&= |E_1| && \text{(since } |\det T||\det T^{-1}| = 1) \\
&= \int_E \chi_{E_1}(y)\,dy.
\end{aligned}
$$

By linearity of the integral, the statement is also true for any simple function $f(x) = \sum_{k=1}^N a_k \chi_{E_k}(x)$, where $E_1, \ldots, E_N$ are measurable.

Write $f = f^+ - f^-$. Since $f^+ \geq 0$, there is an increasing sequence of measurable simple functions $f_k \nearrow f^+$. Then by Monotone Convergence Theorem,

$$\int_E f^+(y)\,dy = |\det T| \int_{T^{-1}E} f^+(Tx)\,dx.$$

Since $f^- \geq 0$, similarly the statement is also true for f^-.

Since $\int_E f(y)\,dy$ exists, at least one of the integrals $\int_E f^+(y)\,dy$, $\int_E f^-(y)\,dy$ is finite (so the case $\infty - \infty$ will not occur), thus we may conclude that

$$
\begin{aligned}
\int_E f(y)\,dy &= \int_E f^+(y)\,dy - \int_E f^-(y)\,dy \\
&= |\det T| \int_{T^{-1}E} f^+(Tx)\,dx - |\det T| \int_{T^{-1}E} f^-(Tx)\,dx \\
&= |\det T| \int_{T^{-1}E} f(Tx)\,dx.
\end{aligned}
$$

5.7 Q21

We will use the following theorem:

Theorem (Theorem 5.11)**.** Let f be nonnegative and measurable on E. Then $\int_E f = 0$ if and only if $f = 0$ a.e. in E.

We have $\int_{\{f\geq 0\}} f = 0$ since $\{f \geq 0\}$ is a measurable subset of E. Thus by Theorem 5.11, $f = 0$ a.e. in $\{f \geq 0\}$.

Next we have $\int_{\{f<0\}} f = 0$ since $\{f < 0\}$ is a measurable set. This implies $\int_{\{f<0\}}(-f) = -0 = 0$. Since $-f$ is nonnegative and measurable on $\{f < 0\}$, this implies $-f = 0$ a.e. in $\{f < 0\}$.

Therefore $f = 0$ a.e. in $E = \{f \geq 0\} \cup \{f < 0\}$.

6 Chapter 6

6.1 Q2

Let $F(x, y) := f(x)$, $G(x, y) := g(y)$ for all $x, y \in \mathbb{R}^n$. Observe that

$$\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n | F(x, y) > \alpha\} = \{x \in \mathbb{R}^n | f(x) > \alpha\} \times \mathbb{R}^n$$

which is measurable by repeated application of Lemma 5.2 which states that $E \times \mathbb{R}$ is measurable for measurable $E \subseteq \mathbb{R}^m$. Thus, $F(x, y)$ is measurable in $\mathbb{R}^n \times \mathbb{R}^n$.

Similarly,

$$\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n | G(x, y) > \alpha\} = \mathbb{R}^n \times \{y \in \mathbb{R}^n | g(y) > \alpha\}$$

is a measurable set. Thus, $G(x, y)$ is measurable in $\mathbb{R}^n \times \mathbb{R}^n$.

Hence $F(x, y)G(x, y) = f(x)g(y)$ is measurable in $\mathbb{R}^n \times \mathbb{R}^n$.

Let E_1 and E_2 be measurable subsets of $\mathbb{R}^n$. Then χ_{E_1} and χ_{E_2} are measurable in $\mathbb{R}^n$. By the earlier part, $\chi_{E_1}(x)\chi_{E_2}(y)$ is measurable in $\mathbb{R}^n \times \mathbb{R}^n$.

Note that $\chi_{E1}(x)\chi_{E_2}(y) = \chi_{E_1\times E_2}(x,y)$, so $E_1 \times E_2$ is measurable in $\mathbb{R}^n \times \mathbb{R}^n$.

$$
\begin{aligned}
|E_1 \times E_2| &= \iint\limits_{\mathbb{R}^n\times\mathbb{R}^n} \chi_{E_1\times E_2}(x,y)\,dx\,dy \\
&= \iint\limits_{\mathbb{R}^n\times\mathbb{R}^n} \chi_{E_1}(x)\chi_{E_2}(y)\,dx\,dy \\
&= \int\limits_{\mathbb{R}^n} \chi_{E_1}(x)\left[\int\limits_{\mathbb{R}^n} \chi_{E_2}(y)\,dy\right]dx \qquad \text{(by Tonelli's Theorem)} \\
&= \int\limits_{\mathbb{R}^n} \chi_{E_1}(x)\,dx \cdot |E_2| \\
&= |E_1||E_2|.
\end{aligned}
$$

6.2 Q4

By Lemma 6.15, $f(x+t)$ and $f(-x+t)$ are both measurable in $\mathbb{R}^2$. By Tonelli's Theorem,

$$
\begin{aligned}
\iint\limits_{[0,1]^2} |f(x+t) - f(-x+t)|\,dt\,dx &= \int_0^1 \left[\int_0^1 |f(x+t) - f(-x+t)|\,dt\right] dx \\
&\le \int_0^1 c\,dx \\
&= c.
\end{aligned}
$$

We quote the following result:

Theorem (Chapter 5 Exercise 20)**.** Let $y = Tx$ be a nonsingular linear transformation of $\mathbb{R}^n$. If $\int_E f(y)\,dy$ exists, then

$$\int_E f(y)\,dy = |\det T| \int_{T^{-1}E} f(Tx)\,dx.$$

Let $T = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ so that $\begin{pmatrix} \xi \\ \eta \end{pmatrix} = T \begin{pmatrix} x \\ t \end{pmatrix}$. Let $E = [0,1]^2$. We compute that $T^{-1}E$ is the convex hull of $\{(0,0), (-\frac{1}{2}, \frac{1}{2}), (0,1), (\frac{1}{2}, \frac{1}{2})\}$.

$$\begin{aligned}
&\iint\limits_{[0,1]^2} |f(\xi) - f(n)| \, d\eta \, d\xi \\
&= |\det T| \iint\limits_{T^{-1}E} |f(x+t) - f(-x+t)| \, dt \, dx \\
&\leq 2 \iint\limits_{[-1,1]\times[0,1]} |f(x+t) - f(-x+t)| \, dt \, dx \qquad \text{(since } T^{-1}E \subseteq [-1,1] \times [0,1]\text{)} \\
&= 4 \iint\limits_{[0,1]^2} |f(x+t) - f(-x+t)| \, dt \, dx \qquad \text{(by periodicity of } f\text{)} \\
&\leq 4c.
\end{aligned}$$

Hence $f(\xi) - f(\eta)$ is integrable over the square $[0,1]^2$. By Q3, we conclude that $f \in L[0,1]$.

6.3 Q5

(a)

$$\begin{aligned}
\int_E f &= |R(f,E)| \qquad \text{(by definition of the integral)} \\
&= \iint\limits_{R(f,E)} dx \, dy \\
&= \int_0^\infty \left[\int_{\{x:(x,y)\in R(f,E)\}} dx \right] dy \qquad \text{(by Tonelli's Theorem)} \\
&= \int_0^\infty |\{x \in E : f(x) \geq y\}| \, dy \\
&= \int_0^\infty \omega(y) \, dy.
\end{aligned}$$

The last equality follows from the fact that $\omega(y)$ is decreasing thus has countably many points of discontinuity, and $\omega(y) = |\{x \in E : f(x) \geq y\}|$ unless y is a point of discontinuity of ω.

(b)

Note that $f^p(x) = \int_0^{f(x)} py^{p-1}\, dy$ for all $x \in E$. Thus

$$\begin{aligned}
\int_E f^p(x)\, dx &= \int_E \int_0^{f(x)} py^{p-1}\, dy\, dx \\
&= \iint\limits_{R(f,E)} py^{p-1}\, dy\, dx && \text{(by Tonelli's Theorem)} \\
&= \int_0^{\infty} \int_{\{x \in E : f(x) \geq y\}} py^{p-1}\, dx\, dy && \text{(by Tonelli's Theorem)} \\
&= p \int_0^{\infty} y^{p-1} \omega(y)\, dy
\end{aligned}$$

since $\omega(y) = |\{x \in E : f(x) \geq y\}|$ almost everywhere.