Analysis Part 3

www.mathtuition88.com

Book: Measure and Integral by Wheeden and Zygmund

4 Chapter 4

4.1 Q3

$(\implies)$ Assume F is measurable.

Let G be an open set in $\mathbb{R}$. Then $G \times \mathbb{R}$ is open in $\mathbb{R}^2$. Thus

$$F^{-1}(G \times \mathbb{R}) = \{x \in \mathbb{R}^n : f(x) \in G \text{ and } g(x) \in \mathbb{R}\} = f^{-1}(G)$$

is measurable.

Hence f is measurable. Similarly, we can show g is measurable.

$(\impliedby)$ Assume both f and g are measurable in $\mathbb{R}^n$.

Let G be a open set in $\mathbb{R}^2$. By definition of product topology,

$$G = \bigcup_{i=1}^{\infty} (A_i \times B_i)$$

where A_i, B_i are open sets in $\mathbb{R}$.

$$
\begin{aligned}
F^{-1}(G) &= \bigcup_{i=1}^{\infty} F^{-1}(A_i \times B_i) \\
&= \bigcup_{i=1}^{\infty} \{x \in \mathbb{R}^n : f(x) \in A_i \text{ and } g(x) \in B_i\} \\
&= \bigcup_{i=1}^{\infty} \left(f^{-1}(A_i) \cap g^{-1}(B_i)\right).
\end{aligned}
$$

Since both $f^{-1}(A_i)$ and $g^{-1}(B_i)$ are measurable, and the collection of measurable sets is a σ-algebra, thus $F^{-1}(G)$ is measurable. Therefore F is measurable.

4.2 Q5

Let F be the Cantor-Lebesgue function. Define

$$f(x) = \inf\{a \in [0, 1] : F(a) = x\}$$

for $x \in [0, 1]$. Then $f(F(x)) = F(f(x)) = x$ for all $x \in C'$, where C' denotes the Cantor set excluding all right end-points of intervals removed[1]. Hence, f is the inverse of F restricted to C'.

We have that $F(C') = [0, 1]$, so $F(C')$ contains a non-measurable set A. Let $Z \subseteq C'$ be a set of measure zero such that $F(Z) = A$. Define $\phi := \chi_Z$, the characteristic function of Z. Note that

$$f^{-1}\phi^{-1}((0, 2)) = f^{-1}(Z) = F(Z) = A.$$

This shows that $\phi(f(x))$ is not measurable.

Note that f is monotone thus measurable. ϕ is measurable since $\{\phi > \alpha\}$ is either the empty set $\emptyset$, Z, or $[0, 1]$, all of which are measurable.

[1] That is, $C' = C \setminus \{\frac{2}{3}, \frac{2}{9}, \frac{8}{9}, \dots\}$. This is needed to make F injective on C', otherwise no inverse would exist.

Show that the same may be true even if f is continuous.

Let $g(x) = x + F(x)$. Note that $g : [0, 1] \to [0, 2]$ is strictly monotone and continuous, thus it has a continuous inverse. Let $f = g^{-1}$.

We claim that $|g(C)| = 1$, where C is the Cantor set. This is because F is constant on every interval in $[0, 1]\backslash C$, so g maps such an interval to an interval of the same length. So $|g([0, 1] \setminus C)| = 1$. Since $|g([0, 1])| = |[0, 2]| = 2$, this proves the claim that $|g(C)| = 1$.

Similarly to before, let $W \subseteq C$ be a set of measure zero such that $g(W) = A$, where A is a non-measurable set. Define $\phi := \chi_W$. Then

$$f^{-1}\phi^{-1}((0, 2)) = f^{-1}(W) = g(W) = A.$$

Thus $\phi(f(x))$ is not measurable.

4.3 Q12

Assume $f(x)$ is continuous at almost every point of $[a, b]$. Let

$$E = \{x \in [a, b] : f \text{ is continuous at } x\}.$$

Let $Z = [a, b] \setminus E$. Then $|Z| = 0$. Note that Z is measurable, and $E = [a, b] \setminus Z$ is measurable too.

For any finite α, we have

$$\{x \in [a, b] : f(x) > \alpha\} = \{x \in E : f(x) > \alpha\} \cup \{x \in Z : f(x) > \alpha\}.$$

Note that $\{x \in E : f(x) > \alpha\}$ is measurable since f is continuous thus measurable on E.

Since $\{x \in Z : f(x) > \alpha\} \subseteq Z$, thus $|\{x \in Z : f(x) > \alpha\}| = 0$. Hence $\{x \in Z : f(x) > \alpha\}$ is also measurable.

Thus $\{x \in [a, b] : f(x) > \alpha\}$ is measurable. This means f is measurable on $[a, b]$.

Generalize this to functions defined in $\mathbb{R}^n$.

Our proof generalizes to show that if $f(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$, is continuous at almost every point of a n-dimensional interval I, then f is measurable on I.

4.4 Q15

Let $\epsilon > 0$. For each $n \in \mathbb{N}$, define

$$E_n := \{x \in E : |f_k(x)| \leq n \text{ for all k}\} = \bigcap_{k=1}^{\infty} \{x \in E : |f_k(x)| \leq n\}.$$

Note that each E_n is measurable since each f_k is measurable.

Since each $M_x < \infty$, we have that $E_n \nearrow E$. By the Monotone Convergence Theorem for measure, $\lim_{n\to\infty} |E_n| = |E| < \infty$. Thus, there exists N such that

$$|E| - |E_N| = |E \setminus E_N| < \epsilon/2.$$

Let F be a closed set contained in E_N such that $|E_N \setminus F| < \epsilon/2$.

Then $|E \setminus F| = |E \setminus E_N| + |E_N \setminus F| < \epsilon$ and $|f_k(x)| \leq N$ for all k and all $x \in F$.

4.5 Q16

$(\implies)$ Assume $f_k \xrightarrow{m} f$ on E. Let $\epsilon > 0$ be given. By definition of convergence in measure,

$$\lim_{k\to\infty} |\{x \in E : |f(x) - f_k(x)| > \epsilon\}| = 0.$$

By definition of limit, there exists K such that $|\{|f - f_k| > \epsilon\}| < \epsilon$ if $k > K$.

$(\impliedby)$ Assume that given any $\epsilon > 0$, there exists K such that $|\{|f - f_k| > \epsilon\}| < \epsilon$ if $k > K$.

Let $\delta > 0$. Given any $\epsilon > 0$, we wish to show that there exists K such that $|\{|f - f_k| > \delta\}| < \epsilon$ if $k > K$. This would imply $\lim_{k\to\infty} |\{|f - f_k| > \delta\}| = 0$, and thus $f_k \xrightarrow{m} f$.

Case 1: $\epsilon \geq \delta$.

By assumption, there exists K such that $|\{|f - f_k| > \delta\}| < \delta \leq \epsilon$ if $k > K$.

Case 2: $\epsilon < \delta$.

Since $\{|f - f_k| > \delta\} \subseteq \{|f - f_k| > \epsilon\}$, thus

$$|\{|f - f_k| > \delta\}| \leq |\{|f - f_k| > \epsilon\}| < \epsilon$$

if $k > K$.

Give an analogous Cauchy criterion: $f_k \xrightarrow{m} f$ on E if and only if given $\epsilon > 0$, there exists K such that $|\{|f_k - f_l| > \epsilon\}| < \epsilon$ if $k, l > K$.

Proof is analogous to the previous part, using Theorem 4.23 (Cauchy criterion for convergence in measure).

4.6 Q17

4.6.1 Addition

Assume that $f_k \xrightarrow{m} f$ and $g_k \xrightarrow{m} g$ on E. Let $\epsilon > 0$. There exists K_1 such that $|\{|f - f_k| > \epsilon/2\}| < \epsilon/2$ if $k > K_1$. Similarly, there exists K_2 such that $|\{|g - g_k| > \epsilon/2\}| < \epsilon/2$ if $k > K_2$.

Since

$$\{|(f + g) - (f_k + g_k)| > \epsilon\} \subseteq \{|f - f_k| > \epsilon/2\} \cup \{|g - g_k| < \epsilon/2\},$$

thus

$$|\{|(f + g) - (f_k + g_k)| > \epsilon\}| < \epsilon/2 + \epsilon/2 = \epsilon$$

if $k > \max\{K_1, K_2\}$.

Ths implies $f_k + g_k \xrightarrow{m} f + g$ on E.

4.6.2 Multiplication

Further assume $|E| < \infty$. Write

$$f_k g_k - fg = (f_k - f)(g_k - g) + f(g_k - g) + g(f_k - f).$$

There exists K_3 such that $|\{|f - f_k| > \sqrt{\epsilon}\}| < \epsilon/2$ if $k > K_3$. There exists K_4 such that $|\{|g - g_k| > \sqrt{\epsilon}\}| < \epsilon/2$ if $k > K_4$.

If $k > \max\{K_3, K_4\}$, then

$$|\{|(f_k - f)(g_k - g)| > \epsilon\}| \leq |\{|f - f_k| > \sqrt{\epsilon}\}| + |\{|g - g_k| > \sqrt{\epsilon}\}| < \epsilon.$$

Thus $(f_k - f)(g_k - g) \xrightarrow{m} 0$.

Applying Question 15 to $\{f\}$, there is a closed $F \subseteq E$ and a finite M such that $|E \setminus F| < \epsilon/2$ and $|f(x)| \leq M$ for all $x \in F$. We may assume $M \neq 0$, as the case $M = 0$ is trivially true[2].

Thus

$$\begin{aligned} |\{|f(g_k - g)| > \epsilon\}| &= |\{x \in F : |f(g_k - g)| > \epsilon\}| + |\{x \in E \setminus F : |f(g_k - g)| > \epsilon\}| \\ &\leq |\{x \in F : |g_k - g| > \epsilon/M\}| + |E \setminus F| \\ &< \epsilon/2 + \epsilon/2 \end{aligned}$$

for sufficiently large k. Thus $f(g_k - g) \xrightarrow{m} 0$. Similarly $g(f_k - f) \xrightarrow{m} 0$.

By our result of "Addition", $f_k g_k - fg \xrightarrow{m} 0$, that is, $f_k g_k \xrightarrow{m} fg$ on E.

4.6.3 Division

Assume in addition that $g_k \to g$ on E, $g \neq 0$ a.e., and $|E| < \infty$. Note that since we have proved "Multiplication", it suffices to show that $1/g_k \xrightarrow{m} 1/g$ on E.

We use Theorem 4.21 (a.e. convergence and $|E| < \infty$ implies convergence in measure). Note that since $g \neq 0$ a.e., $1/g$ is measurable and finite a.e. in

[2] $|\{|f(g_k - g)| > \epsilon\}| = |\{x \in F : |f(g_k - g)| > \epsilon\}| + |\{x \in E \setminus F : |f(g_k - g)| > \epsilon\}| < 0 + \epsilon/2.$

E. Since $g_k \to g$ on E, for sufficiently large k, $g_k \neq 0$ a.e. so that $1/g_k$ is also measurable and finite a.e. in E.

By Theorem 4.21, since $1/g_k \to 1/g$ a.e. on E and $|E| < \infty$, then $1/g_k \xrightarrow{m} 1/g$ on E.

4.7 Q18

Since $f_k \nearrow f$, we have

$$\omega_{f_k} = |\{f_k > a\}| \leq |\{f_{k+1} > a\}| = \omega_{f_{k+1}}.$$

Lemma 4.1. $\bigcup_{k=1}^{\infty} \{f_k > a\} = \{f > a\}$.

Proof. $\bigcup_{k=1}^{\infty} \{f_k > a\} \subseteq \{f > a\}$ is clear since $f_k \nearrow f$. Let $x \in \{f > a\}$. Since $f_k \nearrow f$, there exists K such that $f(x) - f_K(x) < f(x) - a$. This implies $f_K(x) > a$, so $x \in \{f_K > a\}$. Hence $\{f > a\} \subseteq \bigcup_{k=1}^{\infty} \{f_k > a\}$. □

Since $\{f_k > a\} \nearrow \bigcup_{k=1}^{\infty} \{f_k > a\} = \{f > a\}$, by MCT for measure, $\omega_{f_k} \nearrow \omega_f$.

Part 2

Now assume $f_k \xrightarrow{m} f$. Let $\epsilon > 0$.

Let $E_k = \{|f_k - f| > \epsilon\}$. For any $\delta > 0$, there exists K such that $|E_k| < \delta$ for all $k \geq K$.

Since $f_k = (f_k - f) + f$,

$$\{f_k > a\} \subseteq E_k \cup \{f > a - \epsilon\}.$$

Thus

$$\omega_{f_k}(a) \leq |E_k| + \omega_f(a - \epsilon) < \delta + \omega_f(a - \epsilon)$$

for all $k \geq K$.

This means $\omega_f(a - \epsilon)$ is an eventual upper bound of $\{\omega_{f_k}(a)\}$, thus $\limsup_{k \to \infty} \omega_{f_k}(a) \leq \omega_f(a - \epsilon)$.

Analogously, since $f = (f - f_k) + f_k$, we have $\{f > a + \epsilon\} \subseteq E_k \cup \{f_k > a\}$. Thus

$$\omega_f(a + \epsilon) \leq |E_k| + \omega_{f_k}(a) < \delta + \omega_{f_k}(a)$$

for all $k \geq K$. Hence $\liminf_{k\to\infty} \omega_{f_k}(a) \geq \omega_f(a + \epsilon)$.

Let a be a point of continuity of ω_f. Taking limits as $\epsilon \to 0$ in the inequalities above, we get

$$\limsup_{k\to\infty} \omega_{f_k}(a) \leq \omega_f(a) \leq \liminf_{k\to\infty} \omega_{f_k}(a).$$

Since $\liminf_{k\to\infty} \omega_{f_k}(a) \leq \limsup_{k\to\infty} \omega_{f_k}(a)$ always holds, we get

$$\lim_{k\to\infty} \omega_{f_k}(a) = \omega_f(a)$$

as desired.

4.8 Q19

We prove that if f is continuous in x for each y, and $f(x, y)$ is measurable for each fixed $x \in [0, 1]$, then f is a measurable function of (x, y). Note that this will answer both questions in Q19, since a continuous function is also a measurable function.

Partition $[0, 1]$ into n subintervals with equal length. Define

$$f_n(x, y) = f(\tfrac{k}{n}, y) \text{ for } x \in [\tfrac{k}{n}, \tfrac{k+1}{n}), k = 0, \ldots, n - 1.$$

Since f is continuous at $\frac{k}{n}$ for each fixed y, given $\epsilon > 0$ there exists δ_y such that whenever $|x - \frac{k}{n}| < \delta_y$, $|f(x, y) - f(\frac{k}{n}, y)| < \epsilon$.

For all $n > \frac{1}{\delta_y}$, if $x \in [\frac{k}{n}, \frac{k+1}{n})$, we have $|x - \frac{k}{n}| \leq \frac{1}{n} < \delta_y$, so that $|f(x, y) - f(\frac{k}{n}, y)| < \epsilon$. This means that $f_n(x, y)$ converges to $f(x, y)$ pointwise.

Lemma 4.2.

$$\{(x, y) \in [0, 1]^2 : f_n(x, y) > \alpha\} = \bigcup_{k=0}^{n-1} \left([\tfrac{k}{n}, \tfrac{k+1}{n}) \times \{y \in [0, 1] : f(\tfrac{k}{n}, y) > \alpha\}\right).$$

Proof. ($\subseteq$) If $(p,q) \in \{f_n(x,y) > \alpha\}$, then $f_n(p,q) = f(\frac{k}{n}, q) > \alpha$ where $p \in [\frac{k}{n}, \frac{k+1}{n})$ for some $0 \le k \le n-1$. Note that $q \in \{y \in [0,1] : f(\frac{k}{n}, y) > \alpha\}$. Thus, $(p,q) \in \bigcup_{k=0}^{n-1} \left([\frac{k}{n}, \frac{k+1}{n}) \times \{y \in [0,1] : f(\frac{k}{n}, y) > \alpha\}\right)$.

($\supseteq$) If $(p,q) \in \bigcup_{k=0}^{n-1} \left([\frac{k}{n}, \frac{k+1}{n}) \times \{y \in [0,1] : f(\frac{k}{n}, y) > \alpha\}\right)$, then $p \in [\frac{k}{n}, \frac{k+1}{n})$ for some $0 \le k \le n-1$, and $f(\frac{k}{n}, q) = f_n(p,q) > \alpha$. Thus $(p,q) \in \{(x,y) \in [0,1]^2 : f_n(x,y) > \alpha\}$. $\square$

Since we assumed $f(x,y)$ is measurable for fixed $x \in [0,1]$, $\{y \in [0,1] : f(\frac{k}{n}, y) > \alpha\}$ is measurable for each k. By the previous lemma, we see that $\{f_n(x,y) > \alpha\}$ is a union of measurable sets thus is a measurable set. Thus each f_n is a measurable function.

Hence, $f = \lim_{n\to\infty} f_n$ is measurable.

5 Chapter 5

5.1 Q4

Let $f \in L(0,1)$. Note that x^k is continuous thus measurable on $(0,1)$. So $x^k f(x)$ is measurable on $(0,1)$, for all $k \in \mathbb{N}$.

For $x \in (0,1)$, $x^k \le 1$, so $x^k f(x) \le f(x)$ for all $k \in \mathbb{N}$. Hence

$$\int_0^1 x^k f(x)\,dx \le \int_0^1 f(x)\,dx < \infty,$$

therefore $x^k f(x) \in L(0,1)$ for all $k \in \mathbb{N}$.

Note that $x^k f \to 0$ a.e. on $(0,1)$. This is because $|f(x)| < \infty$ a.e. since $f \in L(0,1)$. Since $|x^k f| \le |f|$ and $|f| \in L(0,1)$, by Lebesgue's Dominated Convergence Theorem,

$$\int_0^1 x^k f(x)\,dx \to \int_0^1 0\,dx = 0.$$

5.2 Q5

Let $\{f_k\}$ be a sequence of measurable functions on E such that $f_k \to f$ a.e. in E. Assume $|E| < \infty$ and there exists $M < \infty$ such that $|f_k| \leq M$ a.e. in E. Note that this implies $|f| \leq M < \infty$ a.e. in E.

By Egorov's Theorem, there is a closed subset $F \subseteq E$ such that $|E \backslash F| < \epsilon$ and $\{f_k\}$ converge uniformly to f on F. Since $|f_k| \leq M$ a.e. on $F \subseteq E$,

$$\int_F |f_k| \leq \int_F M < \infty,$$

so $f_k \in L(F)$ for each k. By Uniform Convergence Theorem, $f \in L(F)$ and $\int_F f_k \to \int_F f$.

Now,

$$\begin{aligned}\left|\int_{E\backslash F} f - \int_{E\backslash F} f_k\right| &\leq \int_{E\backslash F} |f - f_k| \\ &\leq \int_{E\backslash F} (|f| + |f_k|) \\ &\leq \int_{E\backslash F} 2M \\ &< 2M\epsilon.\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $\int_{E\backslash F} f_k \to \int_{E\backslash F} f$. Thus

$$\int_E f_k = \int_F f_k + \int_{E\backslash F} f_k \to \int_F f + \int_{E\backslash F} f = \int_E f$$

as $k \to \infty$.

5.3 Q6

By definition of derivative,

$$\frac{\partial f}{\partial x}(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}.$$

Let h_n be a sequence tending to zero[3]. Define

$$\phi_n(x,y) = \frac{f(x+h_n,y) - f(x,y)}{h_n}.$$

Since for each x, $f(x,y)$ is a measurable function of y, so is $\phi_n(x,y)$. Thus $\frac{\partial f}{\partial x}(x,y) = \lim_{n\to\infty} \phi_n(x,y)$ is a measurable function of y for each x.

By Mean Value Theorem, for each n,

$$\phi_n(x,y) = \frac{\partial f}{\partial x}(c,y) \text{ for some } c \in (x, x+h_n).$$

Thus for all n,

$$|\phi_n(x,y)| \le \sup_{x\in[0,1]} \left| \frac{\partial f}{\partial x}(x,y) \right| < \infty.$$

Since $\frac{\partial f}{\partial x}$ is a bounded function,

$$\int_0^1 \sup_{x\in[0,1]} \left| \frac{\partial f}{\partial x}(x,y) \right| dy < \infty.$$

By Lebesgue's DCT,

$$\lim_{n\to\infty} \int_0^1 \phi_n(x,y)\,dy = \int_0^1 \frac{\partial}{\partial x} f(x,y)\,dy.$$

Note that

$$\begin{aligned}\lim_{n\to\infty} \int_0^1 \phi_n(x,y)\,dy &= \lim_{n\to\infty} \frac{\int_0^1 f(x+h_n,y)\,dy - \int_0^1 f(x,y)\,dy}{h_n} \\ &= \frac{d}{dx} \int_0^1 f(x,y)\,dy.\end{aligned}$$

Thus

$$\frac{d}{dx} \int_0^1 f(x,y)\,dy = \int_0^1 \frac{\partial}{\partial x} f(x,y)\,dy$$

as desired.

[3] with $h_n \neq 0$ and $x + h_n \in [0,1]$.

5.4 Q9

Let $p > 0$ and $\int_E |f - f_k|^p \to 0$ as $k \to \infty$. First we prove a lemma.

Lemma 5.1 (L^p version of Tchebyshev's inequality)**.**

$$|\{f > \alpha\}| \leq \frac{1}{\alpha^p} \int_{\{f>\alpha\}} f^p$$

for $\alpha > 0$, $0 < p < \infty$.

Proof.

$$\int_{\{f>\alpha\}} f^p \geq \int_{\{f>\alpha\}} \alpha^p = \alpha^p |\{f > \alpha\}|.$$

□

Let $\delta > 0$. There exists K such that if $k \geq K$, $\int_E |f - f_k|^p < \delta$. Using the Lemma, for any $\epsilon > 0$, if $k \geq K$ we have

$$|\{|f - f_k| > \epsilon\}| \leq \frac{1}{\epsilon^p} \int_{\{|f-f_k|>\epsilon\}} |f - f_k|^p \leq \frac{1}{\epsilon^p} \int_E |f - f_k|^p < \frac{\delta}{\epsilon^p}.$$

Since $\delta > 0$ is arbitrary, $\lim_{k\to\infty} |\{x \in E : |f(x) - f_k(x)| > \epsilon\}| = 0$, so $f_k \xrightarrow{m} f$ on E.

In particular, there is a subsequence $f_{k_j} \to f$ a.e. in E.