Analysis Part 2

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Book: Measure and Integral by Wheeden and Zygmund

3 Chapter 3

3.1 Q1

Choose c_1 to be the largest integer such that $c_1 b^{-1} \leq x$. Inductively, choose c_n to be the largest integer such that $\sum_{k=1}^{n} c_k b^{-k} \leq x$.

Taking limits as $n \to \infty$, we get

$$\sum_{k=1}^{\infty} c_k b^{-k} \leq x.$$

By the maximality of c_n, we have

$$\sum_{k=1}^{n} c_k b^{-k} \leq x < \sum_{k=1}^{n-1} c_k b^{-k} + (c_n + 1) b^{-n}$$

for all n. So

$$x - \sum_{k=1}^{n} c_k b^{-k} < (c_n + 1) b^{-n} - c_n b^{-n} = b^{-n}$$

for all n.

Taking limits as $n \to \infty$, we get

$$x \leq \sum_{k=1}^{\infty} c_k b^{-k}.$$

Uniqueness:

Suppose $x = \sum_{k=1}^{\infty} c_k b^{-k} = \sum_{k=1}^{\infty} d_k b^{-k}$. Let n be the least index where $c_k \neq d_k$. That is, $c_n \neq d_n$ and $c_k = d_k$ for all $k < n$.

We have

$$(c_n - d_n)b^{-n} = \sum_{k=n+1}^{\infty} d_k b^{-k} - \sum_{k=n+1}^{\infty} c_k b^{-k} = \sum_{k=n+1}^{\infty} (d_k - c_k)b^{-k}.$$

Note that

$$b^{-n} \leq |(c_n - d_n)b^{-n}| = \sum_{k=n+1}^{\infty} |d_k - c_k| b^{-k} \leq \sum_{k=n+1}^{\infty} (b-1)b^{-k} = b^{-n}$$

with strict inequality if there exists $|d_m - c_m| \neq b - 1$ for some $m \geq n + 1$.

Case 1) There exists $|d_m - c_m| \neq b - 1$. Then $b^{-n} < b^{-n}$ is a contradiction so in fact no such index n exists. Hence the expansion is unique.

Case 2) $|d_k - c_k| = b - 1$ for all $k \geq n + 1$. The only possibility is $d_k = b - 1$ and $c_k = 0$ (or vice versa) for all $k \geq n + 1$. So

$$x = \sum_{k=1}^{n} c_k b^{-k} + 0 = (c_1 b^{n-1} + c_2 b^{n-2} + \cdots + c_n)b^{-n},$$

where $c := c_1 b^{n-1} + c_2 b^{n-2} + \cdots + c_n \in \{1, \ldots, b^n - 1\}$ since in this case $0 < x < 1$.

The other expansion is

$$x = \sum_{k=1}^{n} d_k b^{-k} + \sum_{k=n+1}^{\infty} (b-1)b^{-k}.$$

3.2 Q2

3.2.1 (a)

For this question, we write $(0.c_1 c_2 \ldots)_3$ to represent $\sum_{k=1}^{\infty} c_k 3^{-k}$. For example, $0.0\dot{2}_3 = (0.0222\ldots)_3 = 0.1_3 = 1/3$.

($\implies$) Suppose $x = (0.c_1c_2\dots)_3 \in C$.

Case 1) x is not an endpoint of any of the intervals removed.

Note that $c_1 \neq 1$ since $(\frac{1}{3}, \frac{2}{3}) = (0.1_3, 0.2_3)$ is removed in the 1st stage. Note that $c_2 \neq 1$ since $(\frac{1}{9}, \frac{2}{9}) = (0.01_3, 0.02_3)$ and $(\frac{7}{9}, \frac{8}{9}) = (0.21_3, 0.22_3)$ are removed in the 2nd stage. Inductively, we can note that $c_k \neq 1$ since $(0.b_1b_2\dots b_{k-1}1_3, 0.b_1b_2\dots b_{k-1}2)_3$, where $b_i \neq 1$, are removed in the kth stage.

Case 2) x is an end point of some interval removed.

If $x = 0.b_1b_2\dots b_{k-1}1_3$, where $b_i \neq 1$, then we write $x = 0.b_1b_2\cdots_{k-1}0\dot{2}_3$. If $x = 0.b_1b_2\dots b_{k-1}2_3$, where $b_i \neq 1$, we are done already.

We have proved that $x \in C$ implies x has some triadic expansion for which every c_k is either 0 or 2.

($\impliedby$) Assume $x = (0.c_1c_2\dots)_3$ has some triadic expansion for which every c_k is either 0 or 2. Let C_k denote the union of the intervals left at the kth stage, so that $C = \bigcap_{k=1}^{\infty} C_k$. Each c_k is either 0 or 2 implies that x is not removed in the kth stage, for all k. So $x \in \bigcap_{k=1}^{\infty} C_k = C$.

3.2.2 (b)

For this question, we are using notation in pg 43 of the textbook.

Let $x \in C$ and $x = \sum_{k=1}^{\infty} c_k 3^{-k}$, where each c_k is either 0 or 2. If $x = 0$, $f(x) = \sum(\frac{1}{2}c_k)2^{-k}$ is clearly true.

Define $x_n = \sum_{k=1}^{n} c_k 3^{-k}$. Note that either $x_n = 0$, or x_n is the endpoint of some open interval I_j^n (jth interval removed, ordered from left to right, at stage n), where $j = \sum_{k=1}^{n}(\frac{1}{2}c_k)2^{n-k}$. See example[1].

Since f_n is continuous,

$$f_n(x_n) = j2^{-n} = \sum_{k=1}^{n}(\frac{1}{2}c_k)2^{-k}$$

[1] For example, if $x_n = 0.22_3$, it is the endpoint of $(0,21_3, 0.22_3) = I_3^2$.

for all n.

Using the fact that $\{f_n\}$ converges uniformly to continuous f, we have

$$f(x) = \lim_{n\to\infty} f_n(x_n) = \sum_{k=1}^{\infty}(\frac{1}{2}c_k)2^{-k}.$$

3.3 Q4

Let C_k denote the union of the intervals left at the kth stage, so that $C = \bigcap_{k=1}^{\infty} C_k$.

Note that

$$|C_k| = (1-\theta)|C_{k-1}|$$

holds for all $k \geq 1$, where $|C_0| = 1$. Since C is covered by the intervals in any C_k, we have

$$|C|_e \leq |C_k| = (1-\theta)^k$$

for all k. Since $0 < 1-\theta < 1$, we see that $|C|_e = 0$.

To show C is perfect, consider $x \in C$. Each C_k is closed, so C is closed. x lies in an interval I_k in C_k for every k. Let $x_k \in C \setminus \{x\}$ be an endpoint of I_k. Then

$$|x - x_k| \leq |C_k| = (1-\theta)^k \to 0$$

as $k \to \infty$.

Thus $\{x_k\}$ is a sequence in $C \setminus \{x\}$ that converges to x, and thus x is a limit point.

3.4 Q5

Let D_k denote the union of the intervals left at the kth stage, so that the resultant set is $D = \bigcap_{k=1}^{\infty} D_k$.

At each stage k, the length of the intervals removed is $2^{k-1}\delta 3^{-k}$. Thus,

$$|D_k| = 1 - \sum_{j=1}^{k} 2^{j-1}\delta 3^{-j}.$$

Since $D_k \searrow D$ and $|D_1| < \infty$, by Monotone Convergence Theorem for measure,

$$|D| = \lim_{k\to\infty} |D_k| = 1 - \delta.$$

Observe that D_1 cannot contain an interval of length greater than $1/2$, since the interval removed is in the *middle*. Inductively, D_k cannot contain an interval of length greater than $1/2^k$. Thus D cannot contain an interval of length greater than $1/2^k$ for all k, so D contains no intervals.

Since each D_k is closed, D is closed. To show D is perfect, consider $x \in D$. x lies in an interval I_k in D_k for every k. Let $x_k \in D \setminus \{x\}$ be an endpoint of I_k. Then

$$|x - x_k| \leq |I_k| \leq 1/2^k \to 0 \text{ as } k \to \infty.$$

Thus $\{x_k\}$ is a sequence in $D \setminus \{x\}$ that converges to x. Hence D is perfect.

3.5 Q9

Let $\epsilon > 0$. Since $\lim_{n\to\infty} \sum_{k=1}^{n} |E_k|_e = \sum_{k=1}^{\infty} |E_k|_e < \infty$, there exists N such that for $n \geq N$,

$$\sum_{k=n+1}^{\infty} |E_k|_e = \sum_{k=1}^{\infty} |E_k|_e - \sum_{k=1}^{n} |E_k|_e < \epsilon.$$

Write $U_j = \bigcup_{k=j}^{\infty} E_k$, so that $\limsup E_k = \bigcap_{j=1}^{\infty} U_j$. Since $\limsup E_k \subseteq U_{N+1}$,

$$|\limsup E_k|_e \leq |U_{N+1}|_e \leq \sum_{k=N+1}^{\infty} |E_k|_e < \epsilon.$$

Since $\epsilon > 0$ is arbitrary, $|\limsup E_k|_e = 0$ so $\limsup E_k$ has measure zero.

Since $\liminf E_k \subseteq \limsup E_k$, $|\liminf E_k|_e = 0$ as well, so $\liminf E_k$ has measure zero as well.

3.6 Q11

Suppose that $|E|_e < +\infty$.

($\Longrightarrow$) Assume E is measurable. Let $\epsilon > 0$. There exists an open set G such that $E \subseteq G$ and $|G \setminus E|_e < \epsilon$.

Since G is open, it can be written as a countable union of nonoverlapping (closed) intervals, say $G = \bigcup_{k=1}^{\infty} I_k$.

We have

$$\sum_{k=1}^{\infty} |I_k| = |\bigcup_{k=1}^{\infty} I_k| = |G| \leq |E|_e + |G \setminus E|_e < \infty.$$

Thus, there exists N such that

$$\left| \bigcup_{k=N+1}^{\infty} I_k \right| = \sum_{k=N+1}^{\infty} |I_k| < \epsilon.$$

Let $S = \bigcup_{k=1}^{N} I_k$, $N_1 = \bigcup_{k=N+1}^{\infty} I_k$, and $N_2 = G \setminus E$.

Then $E = (S \cup N_1) \setminus N_2$ with $|N_1|_e, |N_2|_e < \epsilon$ as desired.

($\Longleftarrow$) Assume $E = (S \cup N_1) \setminus N_2$. Since S is measurable, we can pick an open set G with $S \subseteq G$ and $|G \setminus S| < \epsilon$. Pick another open set G_1 with $N_1 \subseteq G_1$ and

$$|G_1| < |N_1|_e + \epsilon < 2\epsilon.$$

Note that $(G \cup G_1) \setminus E \subseteq (G \setminus S) \cup G_1 \cup N_2$.

Thus

$$|(G \cup G_1) \setminus E|_e \leq |G \setminus S|_e + |G_1| + |N_2|_e < \epsilon + 2\epsilon + \epsilon = 4\epsilon.$$

Since $G \cup G_1$ is open, and $E \subseteq G \cup G_1$, this means that E is measurable.

3.7 Q12

Lemma 3.1. If $A \subseteq \mathbb{R}$ and $Z \subseteq \mathbb{R}$ with $|Z| = 0$, then $|A \times Z| = 0$. Similarly, $|Z \times A| = 0$.

Proof. Let $\epsilon > 0$. Since $|Z| = 0$, there exists intervals $\{I_k\}$ such that $Z \subseteq \bigcup_{k=1}^{\infty} I_k$ and $\sum_{k=1}^{\infty} |I_k| < \epsilon$.

Write $A_n = A \cap [-n, n]$. Then $A = \bigcup_{n=1}^{\infty} A_n$. Note that

$$A_n \times Z \subseteq [-n, n] \times \bigcup_{k=1}^{\infty} I_k = \bigcup_{k=1}^{\infty} ([-n, n] \times I_k)$$

so

$$|A_n \times Z|_e \leq \sum_{k=1}^{\infty} 2n|I_k| = 2n\epsilon.$$

Since $\epsilon > 0$ is arbitrary, $|A_n \times Z| = 0$ for each n. So

$$|A \times Z|_e = |\bigcup_{n=1}^{\infty} (A_n \times Z)|_e \leq \sum_{n=1}^{\infty} |A_n \times Z|_e = 0.$$

Thus $|A \times Z| = 0$ as desired. $\square$

Now, since E_1 is measurable, $E_1 = H_1 \cup Z_1$, where H_1 is of type F_σ and $|Z_1| = 0$. Similarly, write $E_2 = H_2 \cup Z_2$ where H_2 is of type F_σ and $|Z_2| = 0$. Then

$$E_1 \times E_2 = (H_1 \times H_2) \cup (H_1 \times Z_2) \cup (Z_1 \times H_2) \cup (Z_1 \times Z_2).$$

Note that $H_1 \times H_2$ is of type F_σ, while the other terms have measure zero by the previous lemma. Thus $E_1 \times E_2$ is measurable.

Case 1) Suppose $|E_1|$ and $|E_2|$ are both finite.

Since E_1, E_2 are measurable, for each $k \in \mathbb{N}$ there are open sets $S_k \supseteq E_1$, $T_k \supseteq E_2$ such that $|S_k \setminus E_1| < 1/k$, $|T_k \setminus E_2| < 1/k$. We may assume $S_{k+1} \subseteq S_k$, $T_{k+1} \subseteq T_k$ (if $S_{k+1} \not\subseteq S_k$, then define $S'_{k+1} = S_{k+1} \cap S_k$ instead).

Since S_k is open, $S_k = \bigcup_{k \in \mathbb{N}} I_i$ for some nonoverlapping closed intervals. Similarly, $T_k = \bigcup_{j \in \mathbb{N}} J_j$ for some nonoverlapping closed intervals.

So

$$\begin{aligned}|S_k \times T_k| &= \left| \bigcup_{(i,j)\in\mathbb{N}\times\mathbb{N}} (I_i \times J_j) \right| \\ &= \sum_{i,j\in\mathbb{N}} |I_i \times J_j| \\ &= \sum_{i,j\in\mathbb{N}} |I_i||J_j| \\ &= (\sum_{i\in\mathbb{N}} |I_i|)(\sum_{j\in\mathbb{N}} |J_j|) \\ &= |S_k||T_k|.\end{aligned}$$

Write $S = \bigcap_{k=1}^{\infty} S_k$, $T = \bigcap_{k=1}^{\infty} T_k$. Then $|S \setminus E_1| = |T \setminus E_2| = 0$.

Hence

$$\begin{aligned}|E_1 \times E_2| &= |S \times T| \\ &= \lim_{k=\infty} |S_k \times T_k| \\ &= \lim_{k\to\infty} |S_k||T_k| \\ &= |E_1||E_2|,\end{aligned}$$

where the second equality follows by MCT for measure, since $S_k \times T_k \searrow S \times T$ and $|S_k \times T_k| < \infty$ for some k since $|E_1|$, $|E_2|$ are both finite. The last equality also follows by MCT for measure.

Case 2) Suppose one of $|E_1|$, $|E_2|$ are infinite.

If $|E_1| = \infty$ and $|E_2| > 0$, then write $E_1^n = E_1 \cap [-n, n]$.

$$\begin{aligned}|E_1 \times E_2| &= \lim_{n\to\infty} |E_1^n \times E_2| \\ &= \lim_{n\to\infty} |E_1^n||E_2| \\ &= |E_1||E_2| \\ &= \infty,\end{aligned}$$

where the first equality follows by MCT for measure, since $E_1^n \times E_2 \nearrow E_1 \times E_2$.

If $|E_1| = \infty$ and $|E_2| = 0$, $|E_1 \times E_2| = 0$ by our first lemma.

3.8 Q17

Let f be the Cantor-Lebesgue function, which is continuous.

Lemma 3.2. $f(C) = [0, 1]$, where C is the Cantor set.

Proof. $f(C) \subseteq [0, 1]$ is clear.

Let $y \in [0, 1]$. Write y in its binary expansion, i.e.

$$y = \sum_{k=1}^{\infty} c_k 2^{-k}$$

where $c_k = 0$ or 1.

Consider $x = \sum_{k=1}^{\infty} (2c_k) 3^{-k}$. Since $2c_k = 0$ or 2, by Exercise 2, $x \in C$. Furthermore $f(x) = \sum_{k=1}^{\infty} c_k 2^{-k} = y$. So $y \in f(C)$. Hence $[0, 1] \subseteq f(C)$. $\square$

Then, we have $|f(C)| = 1$. Since any set in $\mathbb{R}$ with positive outer measure contains a non-measurable set, $f(C)$ contains a non-measurable set A.

Note that $f^{-1}(A) \subseteq C$ so $|f^{-1}(A)| = 0$. In particular $f^{-1}(A)$ is measurable. So

$$f(f^{-1}(A)) = A$$

gives the desired counterexample.

3.9 Q20

Let E be a nonmeasurable subset of $[0, 1]$ whose rational translates are disjoint. Consider the translates of E by all rational numbers r, $0 < r < 1$, denoted $E_r = \{x + r : x \in E\}$.

Note that $|\bigcup_r E_r|_e \leq 2$ since $\bigcup_r E_r \subseteq [0, 2]$.

Note that $|E_r|_e = |E|_e$ by Exercise 18, furthermore $|E|_e > 0$ since E is nonmeasurable. So $\sum_r |E_r|_e = \infty$.

Thus the inequality is strict.

3.10 Q23

Let $Z \subseteq \mathbb{R}$ with $|Z| = 0$. Write $Z_n = Z \cap [-n, n]$, then $Z = \bigcup_{n=1}^{\infty} Z_n$. Clearly $|Z_n| = 0$ for each n.

Thus there exists intervals $\{I_k\}$ (depending on n) such that $Z_n \subseteq \bigcup_{k=1}^{\infty} I_k$ and $\sum_{k=1}^{\infty} |I_k| < \epsilon$. We may assume each $I_k \subseteq [-n, n]$.

For each $x \in I_k = [a_k, b_k]$, we have

$$\begin{cases} a_k^2 \leq x^2 \leq b_k^2 & \text{if } 0 \leq a_k \leq b_k \\ 0 \leq x^2 \leq \max\{a_k^2, b_k^2\} & \text{if } a_k \leq 0 \leq b_k \\ b_k^2 \leq x^2 \leq a_k^2 & \text{if } a_k \leq b_k \leq 0. \end{cases}$$

If $a_k \leq 0 \leq b_k$, we discard $I_k = [a_k, b_k]$ and replace it with two intervals $[a_k, 0] \cup [0, b_k]$ instead[2]. Thus we may assume $0 \leq a_k \leq b_k$ or $a_k \leq b_k \leq 0$ for all $I_k = [a_k, b_k]$.

Thus, $x^2 \in J_k$ where J_k is an interval with

$$|J_k| = |a_k^2 - b_k^2| = |a_k - b_k||a_k + b_k| \leq |I_k|(|a_k| + |b_k|) \leq |I_k|(2n).$$

[2]Note that this will not affect $Z_n \subseteq \bigcup_{k=1}^{\infty} I_k$ and $\sum_{k=1}^{\infty} |I_k| < \epsilon$.

So

$$
\begin{aligned}
|\{x^2 : x \in Z_n\}|_e &\leq |\bigcup_{k=1}^{\infty} J_k|_e \\
&\leq \sum_{k=1}^{\infty} |J_k| \\
&\leq 2n \sum_{k=1}^{\infty} |I_k| \\
&< 2n\epsilon.
\end{aligned}
$$

Since $\epsilon > 0$ is arbitrary, $|\{x^2 : x \in Z_n\}|_e = 0$, for all n. Note that $\{x^2 : x \in Z\} = \bigcup_{n=1}^{\infty}\{x^2 : x \in Z_n\}$, so

$$|\{x^2 : x \in Z\}|_e \leq \sum_{n=1}^{\infty} |\{x^2 : x \in Z_n\}|_e = 0.$$