Analysis Part 1

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Book: Measure and Integral by Wheeden and Zygmund

1 Chapter 1

1.1 Q1(q)

Let $\{Tx_k\}$ be a sequence of points of TE. Since E is compact, $\{x_k\}$ has a subsequence $\{x_{k_l}\}$ that converges to some $x \in E$.

By continuity of T,

$$\lim_{l \to \infty} Tx_{k_l} = Tx \in TE.$$

Thus $\{Tx_{k_l}\}$ is a subsequence of $\{Tx_k\}$ that converges to $Tx \in TE$, so TE is compact.

1.2 Q1(r)

We are using notation in the book (pg 13), so $R_\Gamma = \sum_{k=1}^N f(\xi_k)v(I_k)$, $U_\Gamma = \sum_{k=1}^N [\sup_{\mathbf{x} \in I_k} f(\mathbf{x})]v(I_k)$, etc.

($\Longrightarrow$) Assume $A = (R) \int_I f(\mathbf{x})\, d\mathbf{x}$ exists. Given $\epsilon > 0$, there exists $\delta > 0$ such that

$$|A - R_\Gamma| < \frac{\epsilon}{2}$$

for any Γ and any chosen $\{\xi_k\}$, provided that $|\Gamma| < \delta$.

By definition of supremum, there exists $\xi_k \in I_k$ such that

$$f(\xi_k) > \sup_{\mathbf{x}\in I_k} f(\mathbf{x}) - \frac{\epsilon}{2v(I)}.$$

Thus

$$\begin{aligned} R_\Gamma &= \sum_{k=1}^{N} f(\xi_k)v(I_k) \\ &> \sum_{k=1}^{N} \sup_{\mathbf{x}\in I_k} f(\mathbf{x})v(I_k) - (\sum_{k=1}^{N} v(I_k))\frac{\epsilon}{2v(I)} \\ &= U_\Gamma - \frac{\epsilon}{2}. \end{aligned}$$

Hence

$$\begin{aligned} U_\Gamma - A &= U_\Gamma - R_\Gamma + R_\Gamma - A \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Note that $\inf_{|\Gamma|<\delta} U_\Gamma = \inf_\Gamma U_\Gamma$. This is clear once we note that any partition Γ has a refinement Γ' with $|\Gamma'| < \delta$, and that $U_\Gamma' \leq U_\Gamma$ for any refinement Γ' of Γ.

This implies $A \geq \inf_\Gamma U_\Gamma - \epsilon$. Since $\epsilon > 0$ is arbitrary, we have $A \geq \inf_\Gamma U_\Gamma$. Similarly, we can show $A \leq \sup_\Gamma L_\Gamma$. Since $L_\Gamma \leq U_\Gamma$, we have $\sup_\Gamma L_\Gamma \leq \inf_\Gamma U_\Gamma$. Combined with the previous inequalities, this gives $\inf_\Gamma U_\Gamma = \sup_\Gamma L_\Gamma = A$.

($\Longleftarrow$) Assume $\inf_\Gamma U_\Gamma = \sup_\Gamma L_\Gamma = A$. By definition of inf and sup, there exists a partition P_1 such that $U_{P_1} < A + \epsilon/4$, and a partition P_2 such that $L_{P_2} > A - \epsilon/4$. Letting P to be the common refinement of P_1, P_2, we get a partition P such that $U_P - L_P < \epsilon/2$.

Denote $P = \{I_k\}_{k=1}^N$.

We want to find a $\delta > 0$ such that $|A - R_\Gamma| < \epsilon$ whenever $|\Gamma| < \delta$. Note that it suffices to show that $U_\Gamma - L_\Gamma < \epsilon$ instead.

Choose

$$\delta = \frac{\epsilon}{4M|\bigcup \partial I_k|},$$

where $|\bigcup \partial I_k|$ denotes the total area of the union of common boundaries of the intervals I_k. For instance in dimension $n = 1$, $|\bigcup \partial I_k| = N - 1$.

Let $\Gamma = \{J_k\}$ be a partition with $|\Gamma| < \delta$. We split the sum according to whether $J_k \subset I_j$ for some j.

$$\begin{aligned}
U_\Gamma - L_\Gamma &= \sum_{J_k \subset I_j \text{ for some } j} (\sup_{J_k} f - \inf_{J_k} f) v(J_k) + \sum_{J_k \not\subset I_j \ \forall j} (\sup_{J_k} f - \inf_{J_k} f) v(J_k) \\
&\leq \sum_{I_j} (\sup_{I_j} f - \inf_{I_j} f) v(I_j) + \sum_{J_k \not\subset I_j \ \forall j} 2M v(J_k) \\
&\leq U_P - L_P + 2M\delta |\bigcup \partial I_k| \\
&< \epsilon/2 + \epsilon/2.
\end{aligned}$$

The second last inequality $\sum_{J_k \not\subset I_j \ \forall j} 2Mv(J_k) \leq 2M\delta|\bigcup \partial I_k|$ is due to the fact that the intervals $J_k \not\subset I_j$ (for all j) are precisely the intervals J_k that intersect the boundary of some I_j so the total volume $\sum_{J_k \not\subset I_j \ \forall j} v(J_k)$ is bounded above by $\delta|\bigcup \partial I_k|$.

1.3 Q4(a)

First we show $\sup_{k \geq j}(-a_k) = -\inf_{k \geq j}(a_k)$ for all j.

By definition of inf, $\inf_{k \geq j}(a_k) \leq a_k$ for all $k \geq j$. Thus $-\inf_{k \geq j}(a_k) \geq -a_k$. Since sup is the least upper bound, $\sup_{k \geq j}(-a_k) \leq -\inf_{k \geq j}(a_k)$. Similarly, by noting that $\sup_{k \geq j}(-a_k) \geq -a_k$ for all $k \geq j$, we can show $\sup_{k \geq j}(-a_k) \geq -\inf_{k \geq j}(a_k)$.

Taking limits as $j \to \infty$, we get

$$\limsup_{k \to \infty}(-a_k) = \lim_{j \to \infty} \sup_{k \geq j}(-a_k) = -\lim_{j \to \infty} \inf_{k \geq j}(a_k) = -\liminf_{k \to \infty}(a_k).$$

Q4(b)

If the right hand side has the form $\infty + \infty$ or $(-\infty) + (-\infty)$, the inequality clearly holds. Assume that the expression on the right does not have the form $\infty + (-\infty)$ or $-\infty + \infty$.

Then, $\sup_{k \geq j}(a_k + b_k) \leq \sup_{k \geq j}(a_k) + \sup_{k \geq j}(b_k)$. Taking limits as $j \to \infty$ gives

$$\limsup_{k \to \infty}(a_k + b_k) \leq \limsup_{k \to \infty}(a_k) + \limsup_{k \to \infty}(b_k).$$

Q4(c)

For $k \geq j$, we have $0 \leq a_k \leq \sup_{k \geq j} a_k < \infty$ and $0 \leq b_k \leq \sup_{k \geq j} b_k < \infty$. Thus, $a_k b_k \leq (\sup_{k \geq j} a_k)(\sup_{k \geq j} b_k)$ for all $k \geq j$. It follows that

$$\sup_{k \geq j}(a_k b_k) \leq (\sup_{k \geq j} a_k)(\sup_{k \geq j} b_k).$$

Note that $\lim_{j \to \infty}(\sup_{k \geq j} a_k)$ exists since $\sup_{k \geq j} a_k$ is a decreasing and bounded sequence. Similarly, $\lim_{j \to \infty}(\sup_{k \geq j} b_k)$ exists.

Taking limits as $j \to \infty$, we get $\limsup_{k \to \infty}(a_k b_k) \leq (\limsup_{k \to \infty} a_k)(\limsup_{k \to \infty} b_k)$.

Q4(d)

Consider $a_k = \{0, 1, 0, 1, \ldots\}$, $b_k = \{1, 0, 1, 0, \ldots\}$. Then $\limsup_{k \to \infty}(a_k + b_k) = 1$, and $\limsup_{k \to \infty} a_k + \limsup_{k \to \infty} b_k = 1+1 = 2$. Also, $\limsup_{k \to \infty}(a_k b_k) = 0$, and $(\limsup_{k \to \infty} a_k)(\limsup_{k \to \infty} b_k) = (1)(1) = 1$.

WLOG, suppose $\{a_k\}$ converges to a. Let $b = \limsup_{k \to \infty} b_k$.

Using Theorem 1.4, there is a subsequence $\{b_{k_j}\}$ of $\{b_k\}$ that converges to b. Then, $a_{k_j} + b_{k_j} \to a + b$ as $j \to \infty$. Let $\epsilon > 0$. There exists K_1 such that $b_k < b + \epsilon/2$ for $K \geq K_1$. There exists K_2 such that $a_k < a + \epsilon/2$ for $k \geq K_2$. Thus if $K = \max\{K_1, K_2\}$,

$$a_k + b_k < a + b + \epsilon$$

for $k \geq K$. By Theorem 1.4, this means

$$\limsup_{k\to\infty} a_k + \limsup_{k\to\infty} b_k = a + b = \limsup_{k\to\infty}(a_k + b_k).$$

For given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $k \geq N$, $a - \epsilon < a_k < a + \epsilon$. For $k \geq j \geq N$, $\sup_{k\geq j}(a_k + b_k) \geq (a - \epsilon)\sup_{k\geq j} b_k$. Taking limits as $j \to \infty$ gives

$$\limsup_{k\to\infty}(a_k b_k) \geq (a - \epsilon)\limsup_{k\to\infty} b_k.$$

Since $\epsilon > 0$ is arbitrary and $\limsup_{k\to\infty} b_k$ is bounded,

$$\limsup_{k\to\infty}(a_k b_k) \geq (\limsup_{k\to\infty} a_k)(\limsup_{k\to\infty} b_k).$$

Along with part (c), this gives equality.

1.4 Q15

($\Longrightarrow$) Assume that bounded f is Riemann integrable on I, and $A = (R)\int_I f(x)\,dx$. By Exercise 1(r), $\inf_\Gamma U_\Gamma = \sup_\Gamma = A$.

There exists a partition Γ_1 such that $U_{\Gamma_1} < A + \epsilon/2$, and a partition Γ_2 such that $L_{\Gamma_2} > A - \epsilon/2$. Let Γ be the common refinement of Γ_1 and Γ_2. Note that $U_\Gamma \leq U_{\Gamma_1}$ and $L_\Gamma \geq L_{\Gamma_2}$. Hence

$$0 \leq U_\Gamma - L_\Gamma \leq U_{\Gamma_1} - L_{\Gamma_2} < A + \epsilon/2 - (A - \epsilon/2) = \epsilon.$$

($\Longleftarrow$) Assume that $0 \leq U_\Gamma - L_\Gamma < \epsilon$ for some partition Γ of I. Then

$$\inf_\Gamma U_\Gamma - \sup_\Gamma L_\Gamma \leq U_\Gamma - L_\Gamma < \epsilon.$$

Since $\epsilon > 0$ is arbitrary, thus $\inf_\Gamma U_\Gamma = \sup_\Gamma L_\Gamma$. By Exercise 1(r), this means that f is integrable.

1.5 Q16

By uniform convergence of $\{f_k\}$, there exists $N \in \mathbb{N}$ such that whenever $k \geq N$,

$$|f_k(x) - f(x)| < \frac{\epsilon}{4v(I)}$$

for all $x \in I$. This also implies that f is bounded.

By Exercise 15, for each k there is a partition $\Gamma_k = \{I_j\}_{j=1}^{N_k}$ of I such that

$$0 \leq U_{\Gamma_k}(f_k) - L_{\Gamma_k}(f_k) < \frac{\epsilon}{2},$$

where $U_{\Gamma_k}(f_k) := \sum_{j=1}^{N_k}[\sup_{x\in I_j} f_k(x)]v(I_j)$, $L_{\Gamma_k}(f_k) := \sum_{j=1}^{N_k}[\inf_{x\in I_j} f_k(x)]v(I_j)$.

When $k \geq N$,

$$\begin{aligned}
U_{\Gamma_k}(f) - L_{\Gamma_k}(f) &\leq \sum_{j=1}^{N_k}[\sup_{x\in I_j} f_k(x) + \frac{\epsilon}{4v(I)}]v(I_j) - \sum_{j=1}^{N_k}[\inf_{x\in I_j} f_k(x) - \frac{\epsilon}{4v(I)}]v(I_j) \\
&= (U_{\Gamma_k}(f_k) + \frac{\epsilon}{4}) - (L_{\Gamma_k}(f_k) - \frac{\epsilon}{4}) \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2}.
\end{aligned}$$

Thus by Exercise 15, f is Riemann integrable on I.

For $k \geq N$, and any partition $\Gamma = \{I_j\}$,

$$\begin{aligned}
|R_\Gamma(f_k) - R_\Gamma(f)| &= |\sum_{j=1}^{N}[f_k(\xi_j) - f(\xi_j)]v(I_j)| \\
&\leq \sum_{j=1}^{N} |f_k(\xi_j) - f(\xi_j)|v(I_j) \\
&< \sum_{j=1}^{N}(\frac{\epsilon}{4v(I)})v(I_j) \\
&= \frac{\epsilon}{4}.
\end{aligned}$$

Since f_k and f are both Riemann integrable, taking limits as $|\Gamma| \to 0$ gives

$$|(R)\int_I f_k(x)\,dx - (R)\int_I f(x)\,dx| \leq \frac{\epsilon}{4}.$$

This shows that $(R)\int_I f_k(x)\,dx \to (R)\int_I f(x)\,dx$.

1.6 Q18

F^c is open, hence it can be written as a countable union of disjoint open intervals.

Write $F^c = \bigcup_{i=1}^{\infty}(a_i, b_i)$, where $a_i < b_i$. We may assume $(a_i, b_i) \neq (-\infty, \infty)$, as that means $F = \emptyset$, for which the statement is trivially true.

Define

$$g(x) = \begin{cases} f(x), & \text{if } x \in F \\ \frac{b_i - x}{b_i - a_i} f(a_i) + \frac{x - a_i}{b_i - a_i} f(b_i), & \text{if } x \notin F \text{ and } x \in (a_i, b_i), \text{with } a_i, b_i \text{ finite} \\ f(b_i), & \text{if } x \in (-\infty, b_i) \\ f(a_i), & \text{if } x \in (a_i, \infty). \end{cases}$$

In short, $g = f$ on F, and is the straight line segment connecting $f(a_i)$ and $f(b_i)$ on (a_i, b_i). Thus g is clearly continuous on $(-\infty, +\infty)$.

If $|f(x)| \leq M$ for $x \in F$, then our constructed g also satisfies $|g(x)| \leq M$ for $-\infty < x < +\infty$.

2 Chapter 2

2.1 Q2

2.1.1 (i)

Let $V[f; a, b] = \sup_\Gamma S_\Gamma \leq M$. Let $x \in (a, b)$.

Consider the partition $\Gamma = \{a, x, b\}$. Then

$$|f(x) - f(a)| \leq |f(x) - f(a)| + |f(b) - f(x)| = S_\Gamma \leq \sup_\Gamma S_\Gamma \leq M.$$

This implies $f(a) - M \leq f(x) \leq f(a) + M$ so f is bounded.

2.1.2 (ii)

For any partition $\Gamma = \{x_0, x_1, \ldots, x_m\}$,

$$S_\Gamma(cf) = \sum_{i=1}^{m} |cf(x_i) - cf(x_{i-1})| = |c| S_\Gamma(f).$$

Taking sup over all partitions Γ, we get

$$V(cf) = |c| V(f) < \infty.$$

$$\begin{aligned} S_\Gamma(f+g) &= \sum_{i=1}^{m} |f(x_i) + g(x_i) - f(x_{i-1}) - g(x_{i-1})| \\ &\leq \sum_{i=1}^{m} |f(x_i) - f(x_{i-1})| + \sum_{i=1}^{m} |g(x_i) - g(x_{i-1})| \\ &= S_\Gamma(f) + S_\Gamma(g). \end{aligned}$$

Thus,

$$V(f+g) \leq V(f) + V(g) < \infty.$$

By part (i), let $|f(x)| \leq M_f$, $|g(x)| \leq M_g$ on $[a, b]$.

$$\begin{aligned} S_\Gamma(fg) &= \sum_{i=1}^{m} |f(x_i)g(x_i) - f(x_{i-1})g(x_i) + f(x_{i-1})g(x_i) - f(x_{i-1})g(x_{i-1})| \\ &\leq \sum_{i=1}^{m} |g(x_i)||f(x_i) - f(x_{i-1})| + \sum_{i=1}^{m} |f(x_{i-1})||g(x_i) - g(x_{i-1})| \\ &\leq M_g S_\Gamma(f) + M_f S_\Gamma(g). \end{aligned}$$

Thus,

$$V(fg) \leq M_g V(f) + M_f V(g) < \infty.$$

Assume there exists $\epsilon > 0$ such that $|g(x)| \geq \epsilon$ for $x \in [a, b]$. Note that $\frac{1}{|g(x)|} \leq \frac{1}{\epsilon}$.

$$
\begin{aligned}
S_\Gamma(\frac{f}{g}) &= \sum_{i=1}^{m} \left| \frac{f(x_i)}{g(x_i)} - \frac{f(x_{i-1})}{g(x_{i-1})} \right| \\
&\leq \sum_{i=1}^{m} \left| \frac{f(x_i)g(x_{i-1}) - f(x_{i-1})g(x_i) - f(x_{i-1})g(x_{i-1}) + f(x_{i-1})g(x_{i-1})}{g(x_i)g(x_{i-1})} \right| \\
&\leq \frac{1}{\epsilon^2} \sum_{i=1}^{m} |g(x_{i-1})||f(x_i) - f(x_{i-1})| + \frac{1}{\epsilon^2} \sum_{i=1}^{m} |f(x_{i-1})||g(x_i) - g(x_{i-1})| \\
&\leq \frac{M_g}{\epsilon^2} S_\Gamma(f) + \frac{M_f}{\epsilon^2} S_\Gamma(g).
\end{aligned}
$$

Hence

$$V(\frac{f}{g}) \leq \frac{M_g}{\epsilon^2} V(f) + \frac{M_f}{\epsilon^2} V(g) < \infty.$$

2.2 Q5

For any $x \in (a, b]$, f is of bounded variation on $(x, b]$. By considering $\Gamma = \{x, b\}$, we see that $|f(b) - f(x)| \leq M$, which implies $|f(x)| \leq |f(b)| + M$.

Next, let $\Gamma = \{a = x_0, x_1, \ldots, x_n = b\}$ be any partition of $[a, b]$. Then

$$
\begin{aligned}
S_\Gamma[a, b] &= |f(x_1) - f(a)| + \sum_{i=2}^{n} |f(x_i) - f(x_{i-1})| \\
&\leq |f(x_1)| + |f(a)| + V[x_1, b] \\
&\leq |f(a)| + |f(b)| + 2M.
\end{aligned}
$$

Thus

$$V[a, b] = \sup_\Gamma [a, b] \leq |f(a)| + |f(b)| + 2M < \infty.$$

No, it is not necessary that $V[f; a, b] \leq M$. Consider

$$f(x) = \begin{cases} 0, & x = a \\ 1, & a < x \leq b. \end{cases}$$

Note that $V[f; a+\epsilon, b] = 0 \leq \frac{1}{2}$ for any $0 < \epsilon < b-a$. However $V[f; a, b] = 1 > \frac{1}{2}$.

The additional assumption that f is right continuous at a will ensure $V[f; a, b] \leq M$.

By right continuity, for any given $\gamma > 0$, there exists $\delta > 0$ such that for all $x < a+\delta$, $|f(x) - f(a)| < \gamma$.

For any partition $\Gamma = \{a = x_0, x_1, \ldots, x_n = b\}$ with $x_1 < a + \delta$,

$$S_\Gamma = |f(x_1) - f(a)| + \sum_{i=2}^{n} |f(x_i) - f(x_{i-1})| < \gamma + V[x_1, b] \leq \gamma + M.$$

For partitions Γ' that do not satisfy $x_1 < a+\delta$, we can always refine it (e.g. by adding the point $a + \delta/2$) so that the refinement Γ'' satisfies $x_1 < a + \delta$. Then $S_{\Gamma'} \leq S_{\Gamma''} \leq \gamma + M$.

Thus

$$V[f; a, b] = \sup_\Gamma S_\Gamma \leq \gamma + M.$$

Since $\gamma > 0$ is arbitrary, $V[f; a, b] \leq M$.