Analysis Theorems

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Contents

1 Multivariable Calculus

1.1 Differentiability in higher dimensions

A function $f : \mathbb{R}^m \to \mathbb{R}^n$ is differentiable at a point x_0 if there exists a linear map $J : \mathbb{R}^m \to \mathbb{R}^n$ such that

$$\lim_{h \to 0} \frac{\|f(x_0 + h) - f(x_0) - J(h)\|_{\mathbb{R}^n}}{\|h\|_{\mathbb{R}^m}} = 0.$$

1.2 Harmonic function

A harmonic function is a twice continuously differentiable function $f : U \to \mathbb{R}$ (where U is an open subset of $\mathbb{R}^n$) which satisfies Laplace's equation

$$\nabla^2 f = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} = 0$$

everywhere on U.

1.3 Divergence Theorem

$$\int_U \nabla \cdot \mathbf{F} \, dV_n = \oint_{\partial U} \mathbf{F} \cdot \mathbf{n} \, dS_{n-1}$$

One can use the general Stoke's Theorem ($\int_\Omega d\omega = \int_{\partial\Omega} \omega$) to equate the n-dimensional volume integral of the divergence of a vector field $\mathbf{F}$ over a region U to the $(n-1)$-dimensional surface integral of $\mathbf{F}$ over the boundary of U.

1.4 Green's Theorem

Let C be a positively oriented, piecewise smooth, simple closed curve in a plane, and let D be the region bounded by C. If L and M are functions of (x, y) defined on an open region containing D and have continuous partial derivatives there, then

$$\oint_C (L \, dx + M \, dy) = \iint_D \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right) dx \, dy.$$

1.5 2D Divergence Theorem (Equivalent to Green's Theorem)

$$\iint_D (\nabla \cdot \mathbf{F})\, dA = \oint_C \mathbf{F} \cdot \hat{\mathbf{n}}\, ds$$

Proof. (2D Divergence Theorem implies Green's Theorem). We have $\hat{t} = \frac{dr}{ds} = (\frac{dx}{ds}, \frac{dy}{ds})$, $\hat{n} = (\frac{dy}{ds}, -\frac{dx}{ds})$. Then

$$\begin{aligned}
\oint_C (L\, dx + M\, dy) &= \oint_C (M, -L) \cdot (dy, -dx) \\
&= \oint_C (M, -L) \cdot \hat{n}\, ds \\
&= \iint_D \nabla \cdot (M, -L)\, dA \quad \text{(By Divergence Theorem)} \\
&= \iint_D \frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\, dA.
\end{aligned}$$

(Green's Theorem implies 2D Divergence Theorem). Write $F(x, y) = (M(x, y), -L(x, y))$. Then

$$\begin{aligned}
\iint_D (\nabla \cdot F)\, dA &= \iint_D \frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\, dA \\
&= \oint_C (L\, dx + M\, dy) \quad \text{(By Green's Theorem)} \\
&= \oint_C (M, -L) \cdot (dy, -dx) \\
&= \oint_C (M, -L) \cdot \hat{n}\, ds \\
&= \oint_C F \cdot \hat{n}\, ds.
\end{aligned}$$

□

2 Real Analysis

2.1 Carathéodory's Criterion

Let λ^* denote the Lebesgue outer measure on $\mathbb{R}^n$, and let $E \subseteq \mathbb{R}^n$. Then E is Lebesgue measurable if and only if $\lambda^*(A) = \lambda^*(A \cap E) + \lambda^*(A \cap E^c)$ for every $A \subseteq \mathbb{R}^n$.

2.2 Abel's Theorem

Let (a_k) be any sequence in $\mathbb{R}$ or $\mathbb{C}$. Let $G_a(z) = \sum_{k=0}^{\infty} a_k z^k$. Suppose that the series $\sum_{k=0}^{\infty} a_k$ converges. Then

$$\lim_{z \to 1^-} G_a(z) = \sum_{k=0}^{\infty} a_k,$$

where z is real, or more generally, lies within any Stolz angle, i.e. a region of the open unit disk where $|1 - z| \leq M(1 - |z|)$ for some M.

2.3 Fatou's Lemma

Let (f_n) be a sequence of nonnegative measurable functions, then

$$\int \liminf_{n \to \infty} f_n \, d\mu \leq \liminf_{n \to \infty} \int f_n \, d\mu.$$

2.4 Lebesgue's Dominated Convergence Theorem

Let $\{f_n\}$ be a sequence of measurable functions. Suppose that $f_n \to f$ pointwise almost everywhere, and that $|f_n| \leq g$ for all n, where g is integrable. Then f is integrable, and

$$\int f \, d\mu = \lim_{n \to \infty} \int f_n \, d\mu.$$

2.5 Chebyshev's/Markov's Inequality (Proof)

If (X, Σ, μ) is a measure space, f is a non-negative measurable extended real-valued function, and $\epsilon > 0$, then

$$\mu(\{x \in X : f(x) \geq \epsilon\}) \leq \frac{1}{\epsilon} \int_X f \, d\mu.$$

Proof. Define

$$s(x) = \begin{cases} \epsilon, & \text{if } f(x) \geq \epsilon \\ 0, & \text{if } f(x) < \epsilon. \end{cases}$$

Then $0 \leq s(x) \leq f(x)$. Thus $\int_X f(x) \, d\mu \geq \int_X s(x) \, d\mu = \epsilon \mu(\{x \in X : f(x) \geq \epsilon\})$. Dividing both sides by $\epsilon > 0$ gives the result. □

2.6 BV functions of one variable

2.6.1 Total variation

The total variation of a real-valued function f, defined on an interval $[a, b]$, is the quantity

$$V_a^b(f) = \sup_{P \in \mathcal{P}} \sum_{i=0}^{n_P - 1} |f(x_{i+1}) - f(x_i)|$$

where the supremum is taken over the set $\mathcal{P} = \{P = \{x_0, \ldots, x_{n_P}\} \mid P \text{ is a partituion of } [a, b]\}$.

2.6.2 BV function

$f \in BV([a, b]) \iff V_a^b(f) < \infty$.

2.6.3 Jordan decomposition of a function

A real function f is of bounded variation in $[a, b]$ iff it can be written as $f = f_1 - f_2$ of two non-decreasing functions on $[a, b]$.

2.7 Borel measurability

A function f is said to be Borel measurable provided its domain E is a Borel set and for each c, the set

$$\{x \in E \mid f(x) > c\} = f^{-1}(c, \infty)$$

is a Borel set.

2.7.1 Borel set

A Borel set is any set in a topological space that can be formed from open sets through the operations of countable union, countable intersection, and relative complement.

2.8 Excision property of measurable sets (Proof)

If A is a measurable set of finite outer measure that is contained in B, then

$$m^*(B \setminus A) = m^*(B) - m^*(A).$$

Proof. By the measurability of A,

$$\begin{aligned} m^*(B) &= m^*(B \cap A) + m^*(B \cap A^c) \\ &= m^*(A) + m^*(B \setminus A). \end{aligned}$$

Since $m^*(A) < \infty$, we have the result. □

2.9 Outer and Inner Approximation of Lebesgue Measurable Sets (Proof)

Let $E \subseteq \mathbb{R}$. Then each of the following four assertions is equivalent to the measurability of E.

2.9.1 (Outer Approximation by Open Sets and G_δ Sets)

(i) For each $\epsilon > 0$, there is an open set G containing E for which $m^*(G \setminus E) < \epsilon$.

(ii) There is a G_δ set G containing E for which $m^*(G \setminus E) = 0$.

2.9.2 (Inner Approximation by Closed Sets and F_σ Sets)

(iii) For each $\epsilon > 0$, there is a closed set F contained in E for which $m^*(E \setminus F) < \epsilon$.

(iv) There is an F_σ set F contained in E for which $m^*(E \setminus F) = 0$.

Proof. (E measurable implies (i)):

Assume E is measurable. Let $\epsilon > 0$. First we consider the case where $m^*(E) < \infty$. By the definition of outer measure, there is a countable collection of open intervals $\{I_k\}_{k=1}^{\infty}$ which covers E and satisfies

$$\sum_{k=1}^{\infty} l(I_k) < m^*(E) + \epsilon.$$

Define $G = \bigcup_{k=1}^{\infty} I_k$. Then G is an open set containing E. By definition of the outer measure of G,

$$m^*(G) \leq \sum_{k=1}^{\infty} l(I_k) < m^*(E) + \epsilon.$$

Since E is measureable and has finite outer measure, by the excision property,

$$m^*(G \setminus E) = m^*(G) - m^*(E) < \epsilon.$$

Now consider the case that $m^*(E) = \infty$. Since $\mathbb{R}$ is σ-finite, E may be expressed as the disjoint union of a countable collection $\{E_k\}_{k=1}^{\infty}$ of measurable sets, each of which has finite outer measure.

By the finite measure case, for each $k \in \mathbb{N}$, there is an open set G_k containing E_k for which $m^*(G_k \setminus E_k) < \epsilon/2^k$. The set $G = \bigcup_{k=1}^{\infty} G_k$ is open, it contains E and

$$G \setminus E = (\bigcup_{k=1}^{\infty} G_k) \setminus E \subseteq \bigcup_{k=1}^{\infty} (G_k \setminus E_k).$$

Therefore

$$\begin{aligned} m^*(G \setminus E) &\leq \sum_{k=1}^{\infty} m^*(G_k \setminus E_k) \\ &< \sum_{k=1}^{\infty} \epsilon/2^k \\ &= \epsilon. \end{aligned}$$

Thus property (i) holds for E.

((i) implies (ii)):

Assume property (i) holds for E. For each $k \in \mathbb{N}$, choose an open set O_k that contains E such that $m^*(O_k \setminus E) < 1/k$. Define $G = \bigcap_{k=1}^{\infty} O_k$. Then G is a G_δ set that contains E. Note that for each k,

$$G \setminus E \subseteq O_k \setminus E.$$

By monotonicity of outer measure,

$$m^*(G \setminus E) \leq m^*(O_k \setminus E) < 1/k.$$

Thus $m^*(G \setminus E) = 0$ and hence (ii) holds.

((ii) $\implies$ E is measurable):

Now assume property (ii) holds for E. Since a set of measure zero is measurable, $G \setminus E$ is measurable. G is a G_δ set and thus measurable. Since measurable sets form a σ-algebra, $E = G \cap (G \setminus E)^c$ is measurable.

((i) $\implies$ (iii)):

Assume condition (i) holds. Note that E^c is measurable iff E is measurable. Thus there exists an open set $G \supseteq E^c$ such that $m^*(G \setminus E^c) < \epsilon$.

Define $F = \mathbb{R} \setminus G$ which is closed. Note that $F \subseteq E$, and $m^*(E \setminus F) = m^*(G \setminus E^c) < \epsilon$.

((iii) $\implies$ (i)):

Similar.

((ii) $\iff$ (iv)):

Similar idea. Note that a set is G_δ iff its complement is F_σ. □

2.10 Lebesgue's Theorem (Monotone functions)

If the function f is monotone on the open interval (a, b), then it is differentiable almost everywhere on (a, b).

2.11 Absolutely Continuous Functions

2.11.1 Definition

A real-valued function f on a closed, bounded interval $[a, b]$ is said to be absolutely continuous on $[a, b]$ provided for each $\epsilon > 0$, there is a $\delta > 0$ such that for every finite disjoint collection $\{(a_k, b_k)\}_{k=1}^n$ of open intervals in (a, b), if

$$\sum_{k=1}^{n} (b_k - a_k) < \delta,$$

then

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)| < \epsilon.$$

2.12 Equivalent Conditions

The following conditions on a real-valued function f on a compact interval $[a, b]$ are equivalent:

(i) f is absolutely continuous;

(ii) f has a derivative f' almost everywhere, the derivative is Lebesgue integrable, and

$$f(x) = f(a) + \int_a^x f'(t)\,dt$$

for all x on $[a, b]$;

(iii) there exists a Lebesgue integrable function g on $[a, b]$ such that

$$f(x) = f(a) + \int_a^x g(t)\,dt$$

for all x on $[a, b]$.

Equivalence between (i) and (iii) is known as the Fundamental Theorem of Lebesgue integral calculus.

2.13 Lusin's Theorem

Informally, "every measurable function is nearly continuous."

(Royden) Let f be a real-valued measurable function on E. Then for each $\epsilon > 0$, there is a continuous function g on $\mathbb{R}$ and a closed set $F \subseteq E$ for which

$$f = g \text{ on } F \text{ (i.e. } f|_F \text{ is continuous)}$$

and

$$m(E \setminus F) < \epsilon.$$

2.14 Egorov's Theorem

Informally, "every convergent sequence of functions is nearly uniformly convergent."

(Royden) Assume $m(E) < \infty$. Let $\{f_n\}$ be a sequence of measurable functions on E that converges pointwise on E to the real-valued function f.

Then for each $\epsilon > 0$, there is a closed set $F \subseteq E$ for which

$$f_n \to f \text{ uniformly on } F \text{ and } m(E \setminus F) < \epsilon.$$

2.15 Tietze Extension Theorem

If X is a normal topological space and

$$f : A \to \mathbb{R}$$

is a continuous map from a closed subset $A \subseteq X$, then there exists a continuous map

$$F : X \to \mathbb{R}$$

with $F(a) = f(a)$ for all a in A.

Moreover, F may be chosen such that $\sup\{|f(a)| : a \in A\} = \sup\{|F(x)| : x \in X\}$, i.e., if f is bounded, F may be chosen to be bounded (with the same bound as f). F is called a continuous extension of f.

2.16 Pasting Lemma (Proof)

Let X, Y be both closed (or both open) subsets of a topological space A such that $A = X \cup Y$, and let B also be a topological space. If both $f|_X : X \to B$ and $f|_Y : Y \to B$ are continuous, then f is continuous.

Proof. Let U be a closed subset of B. Then $f^{-1}(U) \cap X$ is closed since it is the preimage of U under the function $f|_X : X \to B$, which is continuous. Similarly, $f^{-1}(U) \cap Y$ is closed. Then, their union $f^{-1}(U)$ is also closed, being a finite union of closed sets. □

2.17 Mertens' Theorem

Let (a_n) and (b_n) be real or complex sequences.

If the series $\sum_{n=0}^{\infty} a_n$ converges to A and $\sum_{n=0}^{\infty} b_n$ converges to B, and at least one of them converges absolutely, then their Cauchy product converges to AB.

3 Complex Analysis

3.1 Schwarz Lemma

Let $D = \{z : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$ centered at the origin and let $f : D \to D$ be a holomorphic map such that $f(0) = 0$.

Then, $|f(z)| \leq |z|$ for all $z \in D$ and $|f'(0)| \leq 1$.

Moreover, if $|f(z)| = |z|$ for some non-zero z or $|f'(0)| = 1$, then $f(z) = az$ for some $a \in \mathbb{C}$ with $|a| = 1$ (i.e. f is a rotation).

Proof. Consider

$$g(z) = \begin{cases} \dfrac{f(z)}{z} & \text{if } z \neq 0, \\ f'(0) & \text{if } z = 0. \end{cases}$$

Since f is analytic, $f(z) = 0 + a_1 z + a_2 z^2 + \ldots$ on D, and $f'(0) = a_1$. Note that $g(z) = a_1 + a_2 z + \ldots$ on D, so g is analytic on D.

Let $D_r = \{z : |z| \leq r\}$ denote the closed disk of radius r centered at the origin. The Maximum Modulus Principle implies that, for $r < 1$, given any $z \in D_r$, there exists z_r on the boundary of D_r such that

$$|g(z)| \leq |g(z_r)| = \frac{|f(z_r)|}{|z_r|} \leq \frac{1}{r}.$$

As $r \to 1$ we get $|g(z)| \leq 1$, thus $|f(z)| \leq |z|$. Thus

$$\begin{aligned} |f'(0)| &= |\lim_{z\to 0} \frac{f(z)}{z}| \\ &= \lim_{z\to 0} |\frac{f(z)}{z}| \end{aligned}$$

$$\leq 1.$$

Moreover, if $|f(z)| = |z|$ for some non-zero $z \in D$ or $|f'(0)| = 1$, then $|g(z)| = 1$ at some point of D. By the Maximum Modulus Principle, $g(z) \equiv a$ where $|a| = 1$. Therefore, $f(z) = az$. □

3.2 Maximum Modulus Principle

Let f be a function holomorphic on some connected open subset D of the complex plane $\mathbb{C}$ and taking complex values. If z_0 is a point in D such that $|f(z_0)| \geq |f(z)|$ for all z in a neighborhood of z_0, then the function f is constant on D.

Informally, the modulus $|f|$ cannot exhibit a true local maximum that is properly within the domain of f.

3.3 Cauchy-Riemann Equations

Let $f(x + iy) = u(x, y) + iv(x, y)$. The Cauchy-Riemann equations are:

$$u_x = v_y$$
$$u_y = -v_x.$$

3.3.1 Alternative Form (Wirtinger Derivative)

The Cauchy-Riemann equations can be written as a single equation

$$\frac{\partial f}{\partial \bar{z}} = 0$$

where

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$$

is the Wirtinger derivative with respect to the conjugate variable.

3.3.2 Goursat's Theorem

Suppose $f = u + iv$ is a complex-valued function which is differentiable as a function $f : \mathbb{R}^2 \to \mathbb{R}^2$. Then f is analytic in an open complex domain Ω iff it satisfies the Cauchy-Riemann equations in the domain.

3.3.3 Looman-Menchoff Theorem

Let Ω be an open set in $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$ a **continuous** function. Suppose the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ exist everywhere but a countable set in Ω. Then f is holomorphic iff it satisfies the Cauchy-Riemann equations.

3.4 Rouche's Theorem

If the complex-valued functions f and g are holomorphic inside and on some closed contour K, with $|g(z)| < |f(z)|$ on K, then f and $f + g$ have the same number of zeroes inside K, where each zero is counted as many times as its multiplicity.

3.4.1 Example

Consider the polynomial $z^5 + 3z^3 + 7$ in the disk $|z| < 2$. Let $g(z) = 3z^3 + 7$, $f(z) = z^5$, then

$$\begin{aligned} |3z^3 + 7| &< 3(8) + 7 \\ &= 31 \\ &< 32 \\ &= |z^5| \quad \text{for every } |z| = 2. \end{aligned}$$

Then $f + g$ has the same number of zeroes as $f(z) = z^5$ in the disk $|z| < 2$, which is exactly 5 zeroes.