# Lax Solution Part 5

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Textbook: Functional Analysis by Peter D. Lax

Exercises: Miscellaneous exercises from various chapters.

## 1 Chapter 6: Hilbert Spaces

### 1.1 Exercise 4

Prove lemma 5.

#### (i)

Let $l$ be a nonzero linear functional on $H$. Define $N := \{x \in H \mid l(x) = 0\}$. $N$ is clearly a linear subspace of $H$. We have $\operatorname{codim}(N) = \dim(H/N)$. By the first isomorphism theorem, $H/N = H/\ker l \cong \mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Therefore $\operatorname{codim}(N) = \dim_{\mathbb{K}}(\mathbb{K}) = 1$.

#### (ii)

Assume $\ker l = \ker m = N$. If $l = m$ are the zero linear functionals, then $l = cm$ holds trivially.

Otherwise, by part (i), we have $\dim(H/N) = 1$. Choose $y \notin N$. Then $N + \text{span}\{y\} = H$. Let $x \in H$, then $x = n + ty$ where $n \in N$, $t \in \mathbb{C}$.

**Case 1:** $t \neq 0$**.** We have

$$\frac{l(x)}{m(x)} = \frac{l(n+ty)}{m(n+ty)} = \frac{tl(y)}{tm(y)} = \frac{l(y)}{m(y)}.$$

Thus $l(x) = \frac{l(y)}{m(y)} m(x)$.

**Case 2:** $t = 0$**.** Then $l(x) = \frac{l(y)}{m(y)} m(x) = 0$ still holds.

### (iii)

Let $l$ be a bounded linear functional. Denote its nullspace by $N$. Let $\{x_n\}$ be a sequence in $N$ that converges to $x \in H$. Then

$$\begin{aligned} l(x) &= l(\lim_{n\to\infty} x_n) \\ &= \lim_{n\to\infty} l(x_n) \quad \text{(since } l \text{ is bounded implies } l \text{ is continuous)} \\ &= 0. \end{aligned}$$

So $x \in N$ and thus $N$ is closed.

# 2 Chapter 8: Duals of Normed Linear Spaces

## 2.1 Exercise 1

Show that $Y^\perp$ is a closed linear subspace of $X'$.

*Proof.* Let $\alpha, \beta \in \mathbb{C}$, $l_1, l_2 \in Y^\perp$, $y \in Y$.

$(\alpha l_1 + \beta l_2)(y) = \alpha l_1(y) + \beta l_2(y) = 0$. Therefore $Y^\perp$ is linear.

Let $(l_n)$ be a sequence in $Y^\perp$ converging to $l \in X'$, i.e. $\|l_n - l\| \to 0$. There exists $N$ such that for all $n \geq N$, $\|l_n - l\| = \sup_{x \leq 1} |(l_n - l)(x)| < \epsilon$.

Let $y \in Y$. For $n \geq N$, we have

$$\begin{aligned} |l(y)| &= \|y\| |l(\frac{y}{\|y\|})| \\ &\leq \|y\| |(l - l_n)(\frac{y}{\|y\|})| + \|y\| |l_n(\frac{y}{\|y\|})| \\ &< \|y\| \epsilon. \end{aligned}$$

Since $\epsilon$ is arbitrary, $l(y) = 0$ for all $y \in Y$. Thus $l \in Y^{\perp}$. □

## 2.2 Exercise 2

Let $Y$ be a closed subspace of a normed linear space $X$. Show that the dual of $(X/Y)$ is isometrically isomorphic with $Y^{\perp}$.

*Proof.* Define $\psi : (X/Y)' \to Y^{\perp}$, $l \mapsto \tilde{l}$, where $\tilde{l}(x) = l(x + Y)$.

We note that $\psi$ is linear.

For surjectivity, note that for any $f \in Y^{\perp}$, we can define $g \in (X/Y)'$ such that $g(x + Y) = f(x)$. Then $f(x) = \tilde{g}(x)$. We have that $g$ is well-defined: if $x_1 + Y = x_2 + Y$, then $f(x_1) - f(x_2) = f(x_1 - x_2) = g(x_1 - x_2 + Y) = g(Y) = 0$.

Finally, we have

$$\begin{aligned} \|l\| &= \sup_{\|x+Y\| \leq 1} |l(x + Y)| \\ &= \sup_{\|x+Y\| \leq 1} |\tilde{l}(x)| \\ &= \sup_{\|x+y\| \leq 1, x \in X, y \in Y} |\tilde{l}(x + y)| \\ &= \sup_{\|z\| \leq 1, z \in X} |\tilde{l}(z)| \\ &= \|\tilde{l}\|. \end{aligned}$$

□

# 3 Chapter 15: Bounded Linear Maps

## 3.1 Exercise 1

$$\begin{aligned}
\|M+K\| &= \sup_{\|x\|=1} \|Mx + Kx\| \\
&\leq \sup_{\|x\|=1} (\|Mx\| + \|Kx\|) \\
&\leq \sup_{\|x\|=1} \|Mx\| + \sup_{\|x\|=1} \|Kx\| \\
&= \|M\| + \|K\|
\end{aligned}$$

## 3.2 Exercise 8

Prove that multiplication of maps is a continuous operation in the strong topology on the unit balls of $\mathscr{L}(X,U)$ and $\mathscr{L}(U,W)$.

*Proof.* Let $\{M_n\}$ be a sequence of maps in the unit ball of $\mathscr{L}(X,U)$ converging strongly to $M$, i.e. $\|M_n x - Mx\| \to 0$, for all $x \in X$.

Let $\{N_n\}$ be a sequence of maps in the unit ball of $\mathscr{L}(U,W)$ converging strongly to $N$, i.e. $\|N_n u - Nu\| \to 0$, for all $u \in U$.

$$\begin{aligned}
\|N_n M_n x - NMx\| &\leq \|N_n M_n x - N_n M x\| + \|N_n Mx - NMx\| \\
&\leq \|N_n\|\|M_n x - Mx\| + \|N_n(Mx) - N(Mx)\| \\
&\leq \|M_n x - Mx\| + \|N_n(Mx) - N(Mx)\| \\
&\to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Thus multiplication is continuous. □

## 3.3 Exercise 10

Show that in a complex Hilbert space $(NM)^* = M^* N^*$.

*Proof.* Let $H$ be a complex Hilbert space. Let $M : H \to H$, $N : H \to H$. Let $x, y \in H$. We have

$$\begin{aligned} \langle x, M^* N^* y \rangle &= \langle Mx, N^* y \rangle \\ &= \langle NMx, y \rangle \\ &= \langle x, (NM)^* y \rangle. \end{aligned}$$

This implies $\langle x, M^* N^* y - (NM)^* y \rangle = 0$ for all $x, y \in H$. Thus $M^* N^* = (NM)^*$. $\square$