Lax Solution Part 4

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Textbook: Functional Analysis by Peter D. Lax

Exercises: Ch 16: Q2–4. Ch 21: Q1, 2, 9, 10. Ch 28: 1, 5, 9, 10.

1 Chapter 16

Exercise 2

Let $h = \chi_{[0,1]}$, the characteristic function of $[0,1]$. We have $\|\chi_{[0,1]}\|_\infty = 1$, so $\chi_{[0,1]} \in L^\infty$. Then,

$$\begin{aligned}(Hh)(x) &= \frac{1}{\pi}\int_0^1 \frac{1}{x-t}\,dt \\ &= \frac{1}{\pi}[-\ln|x-t|]_0^1 \\ &= \frac{1}{\pi}\ln\frac{|x|}{|x-1|}.\end{aligned}$$

As $x \to 1$, $(Hh)(x) \to \infty$. Thus, Hh is an unbounded function, so H is not bounded as a map: $L^\infty \to L^\infty$.

Suppose to the contrary H is bounded as a map: $L_1 \to L_1$. In the textbook (pg 183) it is proved that the norm of $H : (L^p)' \to (L^p)'$ is equal to the norm of $H : L^p \to L^p$. Since $(L^1)' \cong L^\infty$, this implies that H is bounded as a map: $L^\infty \to L^\infty$ which is a contradiction.

Thus H is not bounded as a map: $L^1 \to L^1$.

Exercise 3

Consider $f(t) = e^{-at}$, where $a > 0$. Note that $f \in L^p(\mathbb{R}_+)$ for all $1 \le p \le \infty$, since $\|f\|_p = (\frac{1}{ap})^{1/p}$ for $1 \le p < \infty$, $\|f\|_\infty = 1$.

$$\begin{aligned}(Lf)(s) &= \int_0^\infty e^{-(a+s)t}\,dt \\ &= \frac{1}{a+s}\end{aligned}$$

For $p > 1$,

$$\begin{aligned}\|Lf\|_{L^p}^p &= \int_0^\infty \frac{1}{(a+s)^p}\,ds \\ &= \frac{a^{-p+1}}{p-1}.\end{aligned}$$

Case 1) If $1 < p < 2$, consider $f_n(t) = e^{-nt}$, i.e. $a = n$. Then,

$$\begin{aligned}\|L\|^p &= \sup_{f \neq 0} \frac{\|Lf\|_p^p}{\|f\|_p^p} \\ &\ge \lim_{n\to\infty} \frac{\|Lf_n\|_p^p}{\|f_n\|_p^p} \\ &= \lim_{n\to\infty} (\frac{n^{-p+1}}{p-1} \cdot np) \\ &= \lim_{n\to\infty} (\frac{p}{p-1} \cdot n^{-p+2}) \\ &= \infty.\end{aligned}$$

Case 2) If $2 < p < \infty$, consider $f_n(t) = e^{-t/n}$, i.e. $a = n^{-1}$. Then similarly,

$$\begin{aligned}\|L\|^p &\ge \lim_{n\to\infty} (\frac{n^{p-1}}{p-1} \cdot n^{-1}p) \\ &= \lim_{n\to\infty} (\frac{p}{p-1} \cdot n^{p-2}) \\ &= \infty\end{aligned}$$

Case 3) If $p = 1,\ f(t) = e^{-at}$,

$$\begin{aligned}\|Lf\|_{L_1} &= \int_0^\infty \frac{1}{a+s}\,ds \\ &= \lim_{r\to\infty} [\ln|a+s|]_0^r \\ &= \lim_{r\to\infty} \ln\frac{|a+r|}{|a|} \\ &= \infty\end{aligned}$$

Case 4) If $p = \infty,\ f(t) = e^{-at}$,

$$\|Lf\|_\infty = \inf\{C \geq 0 \mid |\frac{1}{a+s}| \leq C \text{ for almost every } s\} = \infty.$$

Thus L is not bounded as a map of $L^p(\mathbb{R}_+) \to L^p(\mathbb{R}_+)$, except for $p = 2$ which is shown in the textbook (pg 183, Theorem 9).

Exercise 4

Let $s = -t$, then

$$g(r) = -\int_0^{-\infty} \frac{f(-s)}{r-s}\,ds = \int_{-\infty}^0 \frac{f(-s)}{r-s}\,ds.$$

Let $h(s) := f(-s)$ for $s \leq 0$, $h(s) = 0$ otherwise. Then,

$$\begin{aligned}g(r) &= \int_{-\infty}^0 \frac{h(s)}{r-s}\,ds \\ &= PV\int \frac{h(s)}{r-s}\,ds \\ &= \pi \cdot \frac{1}{\pi} PV\int \frac{h(s)}{r-s}\,ds \\ &= \pi(Hh)(r)\end{aligned}$$

where H is the Hilbert transform, which is proved to be a bounded map of $L^p(\mathbb{R}) \to L^p(\mathbb{R})$ for all $1 < p < \infty$. Thus,

$$\begin{aligned}
\|L^2 f\|_{L^p(\mathbb{R}_+)} &= \|g\|_{L^p(\mathbb{R}_+)} \\
&= \|\pi H h\|_{L^p(\mathbb{R})} \\
&\leq \pi \|H\| \|h\|_{L^p(\mathbb{R})} \\
&= \pi \|H\| \|f\|_{L^p(\mathbb{R}_+)}
\end{aligned}$$

since

$$\begin{aligned}
\|h\|_{L^p(\mathbb{R})} &= (\int_{-\infty}^{\infty} |h(s)|^p \, ds)^{1/p} \\
&= (\int_{-\infty}^{0} |f(-s)|^p \, ds)^{1/p} \\
&= (\int_{0}^{\infty} |f(t)|^p \, dt)^{1/p} \\
&= \|f\|_{L^p(\mathbb{R}_+)}.
\end{aligned}$$

2 Chapter 21

Exercise 1

(c)

Let $(x_n + y_n)$ be a sequence in $C_1 + C_2$, where $x_n \in C_1$, $y_n \in C_2$. Since C_1 is precompact, (x_n) has a Cauchy subsequence (x_{n_k}). Consider (y_{n_k}), which is a sequence in C_2, thus has a Cauchy subsequence $(y_{n_{k_l}})$. Then $(x_{n_{k_l}} + y_{n_{k_l}})$ is a Cauchy subsequence of $(x_n + y_n)$, since it is the sum of two Cauchy sequences, as shown below.

Lemma 2.1. If (z_n) and (w_n) are Cauchy sequences in a normed linear space X, then $(z_n + w_n)$ is Cauchy.

Proof. $\|z_n + w_n - (z_m + w_m)\| \leq \|z_n - z_m\| + \|w_n - w_m\| \to 0$ as $n, m \to \infty$. □

Method 2:

Let C_1 and C_2 be precompact subsets of a Banach space X. Since C_1 is precompact, $C_1 \subseteq \bigcup_{i=1}^{k} B_{\epsilon/2}(x_i)$, where $x_i \in X$. Similarly, $C_2 \subseteq \bigcup_{j=1}^{n} B_{\epsilon/2}(y_j)$, where $y_j \in X$.

Lemma 2.2.

$$C_1 + C_2 \subseteq \bigcup_{j=1}^{n} \bigcup_{i=1}^{k} B_\epsilon(x_i + y_j).$$

Proof. Let $c_1 + c_2 \in C_1 + C_2$. We have $c_1 \in B_{\epsilon/2}(x_i)$ for some x_i, i.e. $\|c_1 - x_i\| < \epsilon/2$. Similarly, $c_2 \in B_{\epsilon/2}(y_j)$ for some y_j, i.e. $\|c_2 - y_j\| < \epsilon/2$. Thus,

$$\|c_1 + c_2 - (x_i + y_j)\| \leq \|c_1 - x_i\| + \|c_2 - y_j\| < \epsilon.$$

Therefore $c_1 + c_2 \in B_\epsilon(x_i + y_j) \subseteq \bigcup_{j=1}^{n} \bigcup_{i=1}^{k} B_\epsilon(x_i + y_j)$. □

Thus, $C_1 + C_2$ is precompact.

(d)

Let C be a precompact set in a Banach space X, and let $\mathrm{Conv}(C)$ denote its convex hull. Since C is precompact, C can be covered by a finite number of balls of radius ϵ, i.e. $C \subseteq \bigcup_{i=1}^{m} B_\epsilon(y_i)$, where $y_i \in X$.

Let $Y := \{y_1, y_2, \ldots, y_m\}$. Consider the continuous map

$$f : \mathbb{R}^m \to X$$
$$(a_1, a_2, \ldots, a_m) \mapsto \sum_{j=1}^{m} a_j y_j.$$

Let $S = \{(a_1, \ldots, a_m) \mid a_j \geq 0, \sum_{j=1}^{m} a_j = 1\}$. S is closed and bounded thus compact in $\mathbb{R}^m$, by the Heine-Borel theorem. Note that $f(S) = \mathrm{Conv}(Y)$, thus $\mathrm{Conv}(Y)$ is compact because continuous functions map compact sets to compact sets. Thus, in particular $\mathrm{Conv}(Y)$ is precompact.

Lemma 2.3. $\text{Conv}(C) \subseteq \text{Conv}(Y) + B_\epsilon(0)$.

Proof. Let $z = \sum_{j=1}^k b_j z_j \in \text{Conv}(C)$, where $b_j \geq 0$, $z_j \in C$, $\sum_{i=1}^k b_j = 1$. We have that $z_j \in B_\epsilon(y_{i(j)})$ for some $y_{i(j)}$ (where $1 \leq i(j) \leq m$ depends on j), i.e. $\|z_j - y_{i(j)}\| < \epsilon$.

Write $z_j = y_{i(j)} + (z_j - y_{i(j)})$. Then, $z = \sum_{j=1}^k b_j y_{i(j)} + \sum_{j=1}^k b_j(z_j - y_{i(j)})$. Note that $\sum_{j=1}^k b_j y_{i(j)} \in \text{Conv}(Y)$.

$$\begin{aligned} \|\sum_{j=1}^k b_j(z_j - y_{i(j)})\| &\leq \sum_{j=1}^k b_j \|z_j - y_{i(j)}\| \\ &< \sum_{j=1}^k b_j \epsilon \\ &= \epsilon \end{aligned}$$

Thus $\sum_{j=1}^k b_j(z_j - y_{i(j)}) \in B_\epsilon(0)$. □

Since $\text{Conv}(Y)$ and $B_\epsilon(0)$ are both precompact, $\text{Conv}(Y) + B_\epsilon(0)$ is precompact. $\text{Conv}(C)$ is a subset of $\text{Conv}(Y) + B_\epsilon(0)$ and thus precompact, since a subset of a precompact set is clearly precompact.

(e)

Let (Mx_n) be a sequence of points of MC. Since (x_n) is a sequence in C, it has a Cauchy subsequence (x_{n_k}).

$$\begin{aligned} \|Mx_{n_k} - Mx_{n_l}\| &= \|M(x_{n_k} - x_{n_l})\| \\ &\leq \|M\| \|x_{n_k} - x_{n_l}\| \\ &\to 0 \quad \text{as } k, l \to \infty. \end{aligned}$$

(Mx_{n_k}) is a Cauchy subsequence of (Mx_n), thus MC is precompact.

Exercise 2

Let $D : X \to U$ be a bounded linear map such that $\dim R_D < \infty$. X and U are Banach spaces.

Let B be the closed unit ball in X. Since R_D is a finite-dimensional subspace of U, it is complete. Hence, we may consider $D : X \to R_D$, where X and R_D are both Banach spaces.

Let $Dx \in DB$. Then, $\|Dx\| \leq \|D\|\|x\| \leq \|D\|$. Thus $DB \subseteq \{u \in R_D \mid \|u\| \leq \|D\|\} := G$.

Thus, the closed ball G is compact since R_D is finite-dimensional. This means G is precompact, thus DB is precompact since $DB \subseteq G$. Hence D is a compact map.

Exercise 9

Counterexample to CM_n tend uniformly to CM:

Proof. Let $X = l_2$. Consider $C : X \to X$, $Cx = (x, e_1)e_1$, i.e. $C(x_1, x_2, \dots) = (x_1, 0, 0, \dots)$. Since $\dim R_C = 1 < \infty$, thus C is compact.

Consider $M_n : X \to X$, $M_n x = (x, e_n)e_1$, i.e. $M_n(x_1, x_2, \dots) = (x_n, 0, 0, \dots)$.

Since $x \in l_2$, $x_n \to 0$. Thus, $\|M_n x - \mathbf{0}x\| \to 0$ as $n \to \infty$, i.e. $\{M_n\}$ tends strongly to $\mathbf{0}$. However,

$$\begin{aligned}
\|CM_n - C\mathbf{0}\| &= \|CM_n\| \\
&= \sup_{\|x\|=1} \|CM_n x\| \\
&\geq \|CM_n(e_n)\| \\
&= \|C(1, 0, 0, \dots)\| \\
&= \|(1, 0, 0, \dots)\| \\
&= 1.
\end{aligned}$$

Thus CM_n does not tend uniformly to CM. □

Proof of M_nC tends uniformly to MC:

Assume $C : X \to U$ is compact and $\{M_n\}$ tends strongly to $M : U \to V$, i.e. $\|M_n u - Mu\| \to 0$ for all $u \in U$. Rewriting, $\|(M_n - M)u - \mathbf{0}u\| \to 0$ for all $u \in U$. This means $\{M_n - M\}$ converges strongly (thus weakly) to the zero operator.

By the Principle of Uniform Boundedness, there exists $c > 0$ such that $\|M_n - M\| \leq c$ for all $n \in \mathbb{N}$.

Fix $n \in \mathbb{N}$. By definition, we have

$$\|M_nC - MC\| = \sup_{\|x\| \leq 1} \|(M_n - M)Cx\|.$$

Let $\{x_k\}$ be a sequence such that $\|x_k\| \leq 1$ and

$$\|(M_n - M)Cx_k\| \geq \|M_nC - MC\| - \frac{1}{k}.$$

Since C is compact, it maps the closed unit ball B to a precompact set in U. Thus $CB \subseteq \bigcup_{i=1}^{N} B_\epsilon(u_i)$. Each $Cx_k \in B\epsilon(u_{i(k)})$ for some $i(k)$ depending on k.

Let $\epsilon > 0$. There is a single $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, $\|M_n u_i - Mu_i\| < \epsilon$ for all $1 \leq i \leq N$.

For $n \geq N_1$, choose $k > 1/\epsilon$. Then,

$$\begin{aligned}\|M_nC - MC\| &\leq \|(M_n - M)Cx_k\| + \frac{1}{k} \\ &\leq \|(M_n - M)(Cx_k - u_{i(k)})\| + \|(M_n - M)u_{i(k)}\| + \frac{1}{k} \\ &< c\epsilon + \epsilon + \epsilon.\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $\|M_nC - MC\| \to 0$ as $n \to \infty$.

Thus M_nC tends uniformly to MC.

Exercise 10

(i)

Let $\{x_n\}$ be a weakly convergent sequence, i.e. $\lim_{n\to\infty} l(x_n) = l(x)$ for every l in X'. By the Principle of Uniform Boundedness, there exists $c > 0$ such that $\|x_n\| \le c$ for all $n \in \mathbb{N}$.

Let $\{Cx_{n_k}\}$ be any subsequence of $\{Cx_n\}$. Since C maps bounded set to precompact set, $\{Cx_{n_k}\}$ has a Cauchy subsequence $\{Cx_{n_{k_l}}\}$. Since U is complete, $\{Cx_{n_{k_l}}\}$ converges strongly to some $u \in U$, i.e.

$$\lim_{l\to\infty} \|Cx_{n_{k_l}} - u\| = 0.$$

Since strong convergence implies weak convergence, $Cx_{n_{k_l}} \rightharpoonup u$. On the other hand, for any $\phi \in U'$, $\lim_{l\to\infty} \phi(Cx_{n_{k_l}}) = \lim_{l\to\infty}(C'\phi)(x_{n_{k_l}}) = (C'\phi)(x) = \phi(Cx)$. By uniqueness of weak limit, $u = Cx$. Thus every subsequence of $\{Cx_n\}$ has a further subsequence that converges to Cx.

Lemma 2.4. Let $\{y_n\}$ be a sequence in a normed linear space. Suppose every subsequence of $\{y_n\}$ has a further subsequence that converges (strongly) to y. Then $\{y_n\}$ converges (strongly) to y.

Proof. Suppose not. Then there exists $\epsilon > 0$ such that $\|y_n - y\| \ge \epsilon$ for infinitely many n. Then there exists a subsequence $\{y_{n_k}\}$ with $\|y_{n_k} - y\| \ge \epsilon$ for all $k \in \mathbb{N}$ so $\{y_{n_k}\}$ has no further subsequence that converges to y. Contradiction. □

Apply Lemma 2.4 to $\{Cx_n\}$, we see that $\{Cx_n\}$ converges strongly to Cx.

(ii)

The converse of theorem 9 is: Let $M : X \to U$ map every weakly convergent sequence into one that converges strongly. Then M is compact.

It is false. Consider the identity operator $I : l^1 \to l^1$. We will prove below that in l^1, every weakly convergent sequence is strongly convergent. Thus I maps every weakly convergent sequence into one that converges strongly.

However, in l^1 the closed unit ball B is not compact since l^1 is infinite-dimensional. Thus $\overline{IB} = \overline{B} = B$ is not compact. Thus I is not a compact operator.

Theorem 2.5. In l^1, every weakly convergent sequence is strongly convergent.

Proof. Let (x^k) be a sequence in l^1. Suppose $x^k \rightharpoonup x$. WLOG, by subtracting x, we may assume that $x^k \rightharpoonup 0$ but $\|x^k\|_1 \geq \epsilon$ for some $\epsilon > 0$. From this assumption we are going to derive a contradiction by constructing a $f \in l^\infty$ with $\|f\|_\infty = 1$ and a subsequence (x^{k_l}) such that $\langle f, x^{k_l} \rangle$ does not converge to zero. (Notation: We write $\langle f, x \rangle := \sum_{m=1}^{\infty} f_m x_m$).

We initialize $j_0 = 0$, set $k_1 = 1$, choose $j_1 \in \mathbb{N}$ such that $\sum_{j>j_1} |x_j^1| \leq \epsilon/6$ and define the first j_1 entries of f as $f_j = \operatorname{sgn}(x_j^1)$ for $1 \leq j \leq j_1$.

Now proceed inductively and assume that for some $l \geq 1$ the numbers $j_1, \ldots, j_l$ the subsequence $\{x^1, \ldots, x^{k_l}\}$ and the entries $f_1, \ldots, f_{j_l}$ have already been constructed and fulfill for all $m \leq l$:

$$|\sum_{j \leq j_{m-1}} f_j x_j^{k_m}| \leq \epsilon/6 \tag{1}$$

$$\sum_{j=j_{m-1}+1}^{j_m} f_j x_j^{k_m} \geq 2\epsilon/3 \tag{2}$$

$$\sum_{j>j_m} |x_j^{k_m}| \leq \epsilon/6 \tag{3}$$

Note that for $l = 1$ these conditions are fulfilled: (1) is fulfilled since the sum is empty, (2) is fulfilled since $\sum_{j=1}^{j_1} f_j x_j^1 = \|x\|_1 - \sum_{j>j_1} |x_j^1| > 5\epsilon/6$ and (3) is fulfilled by definition. To go from a given l to the next one, we first observe that $x^k \rightharpoonup 0$ implies that for all j it holds that $x_j^k \to 0$. Hence, we may take k_{l+1} such that $\sum_{j \le j_l} |x_j^{k_l+1}| \le \epsilon/6$ and take $k_{l+1} > k_l$.

Since $x^{k_{l+1}}$ is a summable sequence, we find j_{l+1} such that $\sum_{j > j_{l+1}} |x_j^{k_l+1}| < \epsilon/6$ and again we may take $j_{l+1} > j_l$. We set $f_j = \operatorname{sgn}(x_j^{k_l+1})$ for $j_l \le j \le j_{l+1}$ and observe $\sum_{j=j_l+1}^{j_{l+1}} f_j x_j^{k_{l+1}} = \sum_{j=j_l+1}^{j_{l+1}} |x_j^{k_l+1}| > \|x^{k_{l+1}}\|_1 - \epsilon/3 > 2\epsilon/3$.

By construction, the properties (1), (2) and (3) are fulfilled for $l+1$, and we continue the procedure ad infinitum. For the resulting $f \in l^\infty$, we have $\|f\|_\infty = 1$, and

$$\begin{aligned}
\langle f, x^{k_l} \rangle &= \sum_j f_j x_j^{k_l} \\
&= \sum_{j \le j_{l-1}} f_j x_j^{k_l} + \sum_{j=j_{l-1}+1}^{j_l} f_j x_j^{k_l} + \sum_{j > j_l} f_j x_j^{k_l} \\
&\ge -|\sum_{j \le j-1} f_j x_j^{k_l}| + \sum_{j=j_{l-1}+1}^{j_l} f_j x_j^{k_l} - \sum_{j > j_l} |x_j^{k_l}| \\
&\ge -\epsilon/6 + 2\epsilon/3 - \epsilon/6 \\
&\ge \epsilon/3.
\end{aligned}$$

□

3 Chapter 28

Exercise 1

Let $A : H \to H$ be a symmetric operator. Let $\{x_n\}$ be a sequence in H such that $x_n \to x$ and $Ax_n \to u$, for some $x, u \in H$. Since strong convergence implies weak convergence, we have $(x_n - x, y) \to 0$ and $(Ax_n - u, y) \to 0$ for all $y \in H$.

$$\begin{aligned}|(Ax-u,y)| &= |(A(x-x_n)+Ax_n-u,y)| \\ &\leq |(A(x-x_n),y)| + |(Ax_n-u,y)| \\ &= |(x-x_n,Ay)| + |(Ax_n-u,y)| \\ &\to 0 \quad \text{as } n\to\infty.\end{aligned}$$

This means $(Ax-u,y)=0$ for all $y \in H$, thus $Ax=u$. Hence A is closed.

It is clear that A is linear, using the symmetric property of A and linearity of the inner product.

By the closed graph theorem, A is continuous and hence bounded.

Exercise 5

Let H be a Hilbert space. Assume $\{x_n\}$ converges weakly to $x \in H$ and $\lim_{n\to\infty}\|x_n\| = \|x\|$. We have that $\langle x_n, y\rangle \to \langle x, y\rangle$ for all $y \in H$. In particular, $\langle x_n, x\rangle \to \|x\|^2$.

$$\begin{aligned}\|x_n-x\|^2 &= \langle x_n-x, x_n-x\rangle \\ &= \langle x_n,x_n\rangle - \langle x_n,x\rangle - \langle x,x_n\rangle + \langle x,x\rangle \\ &= \|x_n\|^2 - 2\text{Re}\langle x_n,x\rangle + \|x\|^2 \\ &\to \|x\|^2 - 2\|x\|^2 + \|x\|^2 \\ &= 0\end{aligned}$$

Thus, x_n converges strongly to x.

Exercise 9

Let U be a unitary map. Note that $\|Ux\| = \|x\|$ implies $\|U\| = \sup_{\|x\|=1} \|Ux\| = \sup_{\|x\|=1} \|x\| = 1$. We also have $\|U^*\| = \|U\| = 1$.

$$\begin{aligned}
\langle U^*Ux - x, U^*Ux - x\rangle &= \|U^*Ux\|^2 - \langle U^*Ux, x\rangle - \langle x, U^*Ux\rangle + \|x\|^2 \\
&= \|U^*Ux\|^2 - \langle Ux, Ux\rangle - \langle Ux, Ux\rangle + \|x\|^2 \\
&= \|U^*Ux\|^2 - 2\|Ux\|^2 + \|x\|^2 \\
&= \|U^*Ux\|^2 - \|Ux\|^2 \\
&\leq \|U^*\|^2\|Ux\|^2 - \|Ux\|^2 \\
&= 0
\end{aligned}$$

By positive definiteness of inner product, $U^*Ux = x$ for all x. Thus $U^*U = I$.

Exercise 10

Lemma 3.1. *C is a compact normal operator.*

Proof. We have $C = U - I$, thus $C^* = U^* - I$. By direct computation we have

$$\begin{aligned}
C^*C &= U^*U - U^* - U + I \\
&= 2I - U^* - U \\
&= CC^*.
\end{aligned}$$

□

We use Corollary 2 which states that every compact normal operator has a complete set of orthonormal eigenvectors. Let $\{z_n\}$ be a complete set of orthonormal eigenvectors of C, with $Cz_n = \alpha_n z_n$.

Then, $Uz_n = z_n + \alpha_n z_n = (1 + \alpha_n)z_n$. So $\{z_n\}$ is a complete set of orthonormal eigenvectors of U.

We have $\|Uz_n\| = |1+\alpha_n|\|z_n\| = \|z_n\|$, thus all eigenvalues $\{1+\alpha_n\}$ have absolute value 1, since $\|z_n\| \neq 0$.