Lax Solution Part 3

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Textbook: Functional Analysis by Peter D. Lax
Exercises: Ch 13: Q1–4. Ch 15: Q2–4, 6, 7, 11.

1 Chapter 13

1.1 Exercise 1

Weak Topology is Hausdorff:

Let x, y be distinct points in X. Since $\|x-y\|>0$, there exists $\epsilon>0$ such that $B_\epsilon(x)$ does not contain y. Since $B_\epsilon(x)$ is convex and open, we can use the Hyperplane Separation Theorem to conclude that there exists $l \in X'$, $c \in \mathbb{R}$ such that

$$l(u) < c \text{ for all } u \text{ in } B_\epsilon(x)$$
$$l(y) = c$$

In particular, there exists $\delta>0$ such that $l(x) < c-\delta < c = l(y)$.

$$x \in \{u \in X : -\infty < l(u) < c-\delta\}$$
$$y \in \{u \in X : c-\delta < l(u) < \infty\}$$

The above two sets are open in the weak topology, and are clearly disjoint. Thus the weak topology is Hausdorff.

(i) Addition is continuous:

Consider $f : X \times X \to X$, $f(x,y) = x+y$. Let $\{x : a < l(x) < b\} = l^{-1}(a,b)$ be open in X. (l^{-1} is preimage of l). Since $\{l^{-1}(a,b)\}$ is a subbasis for X, it suffices to check that $f^{-1}(l^{-1}(a,b))$ is open in $X \times X$.

$$\begin{aligned}
f^{-1}(l^{-1}(a,b)) &= \{(x,y) \in X \times X \mid f(x,y) = x+y \in l^{-1}(a,b)\} \\
&= \{(x,y) \mid a < l(x+y) < b\} \\
&= \{(x,y) \mid a < l(x)+l(y) < b\} \\
&= \{(x,y) \mid a - l(y) < l(x) < b - l(y), a - l(x) < l(y) < b - l(x)\} \\
&= \bigcup_i (U_i \times V_i)
\end{aligned}$$

where

$$\begin{aligned}
U_i &= \{x \in X \mid m_i < l(x) < M_i\} \\
V_i &= \{y \in X \mid m_i' < l(y) < M_i'\},
\end{aligned}$$

such that $a \le m_i + m_i' < l(x) + l(y) < M_i + M_i' \le b$ for $x \in U_i$, $y \in V_i$. Each U_i, V_i is open in X, thus $U_i \times V_i$ is open in $X \times X$. Therefore f is continuous.

Multiplication by scalars is continuous:

Consider $g : \mathbb{R} \times X \to X$, $g(k,x) = kx$.

$$\begin{aligned}
g^{-1}(l^{-1}(a,b)) &= \{(k,x) \in \mathbb{R} \times X \mid g(k,x) = kx \in l^{-1}(a,b)\} \\
&= \{(k,x) \mid a < l(kx) < b\} \\
&= \{(k,x) \mid a < kl(x) < b\} \\
&= \{(k,x) \mid \frac{a}{|k|} < \operatorname{sgn}(k) l(x) < \frac{b}{|k|}, \frac{a}{|l(x)|} < k \cdot \operatorname{sgn}(l(x)) < \frac{b}{|l(x)|}\} \\
&= \bigcup_i (U_i \times V_i)
\end{aligned}$$

where

$$\begin{aligned}
U_i &= \{k \in \mathbb{R} \mid m_i < k < M_i\} \\
V_i &= \{x \in X \mid m_i' < l(x) < M_i'\}
\end{aligned}$$

where $m_i, M_i, m_i', M_i' \in \mathbb{R} \cup \{-\infty, \infty\}$ such that $a < kl(x) < b$ for $k \in U_i$, $x \in V_i$.

Each U_i is open in $\mathbb{R}$, each V_i is open in X, thus $U_i \times V_i$ is open in $\mathbb{R} \times X$. Therefore g is continuous.

(iii)

Let U be an open set containing the origin. U is the union of finite intersections of sets of the form $\{x : a < l(x) < b\}$. 0 is in one of the finite intersections, say

$$0 \in \bigcap_{i=1}^{k} \{x : a_i < l_i(x) < b_i\} \subseteq U.$$

$C := \bigcap_{i=1}^{k} \{x : a_i < l_i(x) < b_i\}$ is a convex open set containing the origin: For any $x, y \in C$, $0 \leq t \leq 1$, $l_i(tx + (1-t)y) = tl_i(x) + (1-t)l_i(y) < tb_i + (1-t)b_i = b_i$. Similarly $l_i(tx + (1-t)y) > a_i$. This holds for all $1 \leq i \leq k$. So $tx + (1-t)y \in C$.

Weak* Topology is Hausdorff:

Let l_1, l_2 be distinct in X'. There exists $x \in X$ such that $l_1(x) \neq l_2(x)$. WLOG suppose $l_1(x) < l_2(x)$.

There exists $r \in \mathbb{R}$ such that $l_1(x) < r < l_2(x)$. The sets $U = \{l \in X' : l(x) < r\}$ and $V = \{l \in X' : l(x) > r\}$ are weak*-open: U is open since it is the preimage $T_x^{-1}\{l(x) : l(x) < r\}$, where $T_x(l) = l(x)$, $T_x \in X''$. Similarly V is open.

Clearly, U and V are disjoint and contain l_1 and l_2 respectively.

(i) Addition is continuous:

Consider $f : X' \times X' \to X'$, $f(l_1 + l_2) = l_1 + l_2$. Let $T_x^{-1}(a, b)$ be open in X', where $T_x \in X''$ and $T_x(l) = l(x)$.

$$\begin{aligned} f^{-1}(T_x^{-1}(a,b)) &= \{(l_1, l_2) \in X' \times X' \mid l_1 + l_2 \in T_x^{-1}(a,b)\} \\ &= \{(l_1, l_2) \mid a < T_x(l_1) + T_x(l_2) < b\} \\ &= \{(l_1, l_2) \mid a < l_1(x) + l_2(x) < b\} \\ &= \bigcup_i (U_i \times V_i) \end{aligned}$$

where

$$\begin{aligned} U_i &= \{l_1 \in X' \mid m_i < l_1(x) < M_i\} = T_x^{-1}(m_i, M_i) \\ V_i &= \{l_2 \in X' \mid m_i' < l_2(x) < M_i'\} = T_x^{-1}(m_i', M_i') \end{aligned}$$

such that $a \leq m_i + m_i' < l_1(x) + l_2(x) < M_i + M_i' \leq b$ for all $l_1 \in U_i$, $l_2 \in V_i$. $\bigcup_i (U_i \times V_i)$ is open in $X' \times X'$, thus f is continuous.

(ii) Multiplication is continuous:

Consider $g : \mathbb{R} \times X' \to X'$, $g(k,l) = kl$.

$$\begin{aligned} g^{-1}(T_x^{-1}(a,b)) &= \{(k,l) \in \mathbb{R} \times X' \mid kl \in T_x^{-1}(a,b)\} \\ &= \{(k,l) \mid a < kT_x(l) < b\} \\ &= \{(k,l) \mid a < kl(x) < b\} \\ &= \bigcup_i (U_i \times V_i) \end{aligned}$$

where

$$\begin{aligned} U_i &= \{k \in \mathbb{R} \mid m_i < k < M_i\} = (m_i, M_i) \\ V_i &= \{l \in X' \mid m_i' < l(x) < M_i'\} = T_x^{-1}(m_i', M_i') \end{aligned}$$

such that $a < kl(x) < b$ for all $k \in U_i$, $l \in V_i$. $\bigcup_i (U_i \times V_i)$ is open in $\mathbb{R} \times X'$, thus g is continuous.

(iii)

Let U be an open set containing the origin. U is the union of finite intersections of sets of the form $T_x^{-1}(a,b)$. 0 is in one of the finite intersections, say $0 \in \bigcap_{i=1}^k T_{x_i}^{-1}(a_i,b_i) \subseteq U$, where $T_{x_i} \in X''$ and $T_{x_i}(l) = l(x_i)$.

$C := \bigcap_{i=1}^k T_{x_i}^{-1}(a_i,b_i)$ is a convex open set containing the origin: For any $l_1, l_2 \in C$, $0 \le t \le 1$, $T_{x_i}(tl_1 + (1-t)l_2) = tT_{x_i}(l_1) + (1-t)T_{x_i}(l_2) < tb_i + (1-t)b_i = b_i$. Similarly $T_{x_i}(tl_1 + (1-t)l_2) > a_i$. So $tl_1 + (1-t)l_2 \in C$.

1.2 Exercise 2

(a)

Hausdorff:

Let x, y be distinct points in X. WLOG, there exists $r \in \mathbb{R}$ such that $l_\alpha(x) < r < l_\alpha(y)$ for some linear functional l_α. Then $l_\alpha^{-1}(-\infty, r)$ and $l_\alpha^{-1}(r, \infty)$ are disjoint open sets that contain x and y respectively.

(i) Addition is continuous:

Consider $f : X \times X \to X$, $f(x,y) = x + y$.

$$\begin{aligned} f^{-1}(l_\alpha^{-1}(a,b)) &= \{(x,y) \in X \times X \mid a < l_\alpha(x+y) < b\} \\ &= \{(x,y) \in X \times X \mid a < l_\alpha(x) + l_\alpha(y) < b\} \end{aligned}$$

Remaining proof similar to weak topology case.

The proof for (ii) Multiplication by scalars is continuous, and (iii) 0 has a convex basis is also similar to weak topology case.

(b)

($\Longleftarrow$) Suppose $l = \sum_{i=1}^{k} c_i l_i$ is a finite linear combination. Let (a, b) be an open interval in $\mathbb{R}$.

$$\begin{aligned} l^{-1}(a,b) &= \{x \in X \mid a < \sum_{i=1}^{k} c_i l_i(x) < b\} \\ &= \bigcup_i \{x \in X \mid m_i < l_i(x) < M_i \text{ for } 1 \leq i \leq k\} \end{aligned}$$

such that $a \leq \sum_{i=1}^{k} c_i m_i < \sum_{i=1}^{k} c_i l_i(x) < \sum_{i=1}^{k} c_i M_i \leq b$.

$$\begin{aligned} \{x \in X \mid m_i < l_i(x) < M_i \text{ for } 1 \leq i \leq k\} &= \bigcap_{i=1}^{k} \{x \in X \mid m_i < l_i(x) < M_i\} \\ &= \bigcap_{i=1}^{k} l_i^{-1}(m_i, M_i) \end{aligned}$$

which is open. Therefore $l^{-1}(a, b)$ is open and thus l is continuous.

($\Longrightarrow$) Assume l is continuous.

Lemma 1.1. There exists $l_1, \ldots, l_n$ such that $\bigcap_{i=1}^{n} \ker(l_i) \subseteq \ker(l)$.

Proof. Consider $l^{-1}(-\epsilon, \epsilon)$ which is open. Since $\{l_{\alpha_i}^{-1}(-\delta_i, \delta_i)\}$ forms a subbasis for the topology, $l^{-1}(-\epsilon, \epsilon)$ is the union of finite intersections $\bigcap_{i=1}^{n} l_i^{-1}(-\delta_i, \delta_i)$.

Let $x \in \bigcap_{i=1}^{n} \ker(l_i)$. Then $l_i(x) = 0$ for all i.

$x \in \bigcap_{i=1}^{n} l_i^{-1}(-\delta_i, \delta_i) \subseteq l^{-1}(-\epsilon, \epsilon)$. Thus $-\epsilon < l(x) < \epsilon$. Since $\epsilon > 0$ is arbitrary, $l(x) = 0$, thus $x \in \ker(l)$. □

Lemma 1.2. If $\bigcap_{i=1}^{n} \ker(l_i) \subseteq \ker(l)$, then l is a finite linear combination of the l_i.

Proof. Consider $F : X \to \mathbb{R}^n$ given by $F(x) = (l_1(x), l_2(x), \ldots, l_n(x))$. We have an induced isomorphism $\widetilde{F} : X/\ker F \to \mathrm{Im}(F)$.

Since $\bigcap_{i=1}^{n} \ker l_i = \ker F \subseteq \ker l$, we have an induced linear functional $\widetilde{l} : X/\ker F \to \mathbb{R}$, and can pull that back to $\mathrm{Im}(F)$ as $\hat{l} := \widetilde{l} \circ \widetilde{F}^{-1}$. We can extend $\hat{l}$ to all of $\mathbb{R}^n$ (extend a basis of $\mathrm{Im}(F)$ to a basis of $\mathbb{R}^n$, and choose arbitrary values, e.g. 0, on the basis vectors not in $\mathrm{Im}(F)$). Thus there is a linear functional $\phi : \mathbb{R}^n \to \mathbb{R}$ with

$$\phi \circ F = \phi\,|_{\mathrm{Im}(F)} \circ F = \hat{l} \circ F = \widetilde{l} \circ \widetilde{F}^{-1} \circ F = \widetilde{l} \circ \pi = l,$$

where $\pi : X \to X/\ker F$ is the canonical projection.

Every linear functional $\mathbb{R}^n \to \mathbb{R}$ can be written as a linear combination of the component projections, so there are $c_1, \ldots, c_n$ with

$$\phi(u_1, u_2, \ldots, u_n) = \sum_{k=1}^{n} c_k u_k,$$

thus this means $l(x) = \phi(F(x)) = \sum_{k=1}^{n} c_k l_k(x)$ for all $x \in X$. Thus $l = \sum_{k=1}^{n} c_k l_k$. □

1.3 Exercise 3

Let $f : X \times Y \to W$ be a continuous function, where $(X, \mathcal{T}_1)$, $(Y, \mathcal{T}_2)$, $(W, \mathcal{T}_3)$ are topological spaces. Define $g : X \to W$, $g(a) = f(a, b_0)$ for fixed $b_0 \in Y$.

Let $a \in X$ and let V be an open set in W containing $g(a) = f(a, b_0)$. Since f is continuous, there is an open set U in $X \times Y$ such that $(a, b_0) \in U$ and $f(U) \subseteq V$. We may take $U = A \times B$, where A is open in X, B is open in Y.

$a \in A$ since $(a, b_0) \in A \times B$. Also, $g(A) = f(A, b_0) \subseteq f(A \times B) = f(U) \subseteq V$. Thus g is continuous.

1.4 Exercise 4

Let K be a convex subset of a LCT linear space X. Let $x, y \in \overline{K}$, and $0 \leq t \leq 1$. Every open set containing x (resp. y) contains a point of K.

Define $w = tx + (1-t)y$. Let W be an arbitrary open set containing w. If $t = 0$, then $w = y \in \overline{K}$. If $t = 1$, then $w = x \in \overline{K}$. Thus we may assume $0 < t < 1$.

$$x = \frac{1}{t}w - \left(\frac{1-t}{t}\right)y \in \frac{1}{t}W - \left(\frac{1-t}{t}\right)y$$

which is open since it is the scaling and translation of W. Thus $\frac{1}{t}w - (\frac{1-t}{t})y$ contains a point $k_1 \in K$. We have $tk_1 + (1-t)y \in W$.

Similarly, $y \in (\frac{1}{1-t})W - (\frac{t}{1-t})k_1$ which is open. Thus $(\frac{1}{1-t})W - (\frac{t}{1-t})k_1$ contains a point k_2 of K. Then $(1-t)k_2 + tk_1 \in W \cap K$.

Thus any open set containing w contains a point in K, hence $w \in \overline{K}$. Therefore $\overline{K}$ is convex.

2 Chapter 15

2.1 Exercise 2

Let $l \in U'$. Note that $M'l \in X'$, thus

$$\begin{aligned}\lim_{n\to\infty} l(Mx_n) &= \lim_{n\to\infty} M'l(x_n) \\ &= M'l(x) \quad (\text{since } x_n \xrightarrow{w} x) \\ &= l(Mx).\end{aligned}$$

Thus Mx_n converges weakly to Mx. Note that this method does not use the fact that U is reflexive.

Method 2:

Let M be a bounded linear map: $X \to U$. Let x_n be a sequence in X weakly convergent to x, i.e. $\lim_{n\to\infty} l(x_n) = l(x)$ for every l in X'. Denote $\hat{u}$ to be the image of u under the canonical embedding of U into U'', where $\hat{u}(f) = f(u)$ for all $f \in U'$.

Lemma 2.1. $M' : U' \to X'$ is weak* continuous, where M' is the transpose of M.

Proof. Let $\hat{x}^{-1}(V)$ be open in X', where V is open in the base field $\mathbb{R}$ or $\mathbb{C}$, and $\hat{x}$ is the image of x under the canonical embedding.

$$\begin{aligned}M'^{-1}(\hat{x}^{-1}(V)) &= \{f \in U' \mid M'(f) \in \hat{x}^{-1}(V)\} \\ &= \{f \in U' \mid fM \in \hat{x}^{-1}(V)\} \\ &= \{f \in U' \mid \hat{x}(fM) \in V\} \\ &= \{f \in U' \mid f(Mx) \in V\} \\ &= \{f \in U' \mid \widehat{Mx}(f) \in V\} \\ &= \widehat{Mx}^{-1}(V)\end{aligned}$$

which is open in U'. □

Corollary 2.2. Similarly, $M'' : X'' \to U''$ is weak* continuous.

Lemma 2.3. $M''\hat{x} = \widehat{Mx}$.

Proof. For any $f \in U'$,

$$\begin{aligned}(M''\hat{x})f &= \hat{x}(M'f)\\ &= M'f(x)\\ &= f(Mx)\\ &= \widehat{Mx}(f).\end{aligned}$$

□

Note that $x_n \xrightarrow{w} x$ implies $\lim_{n\to\infty} \hat{x_n}(l) = \hat{x}(l)$ for all $l \in X'$. This means $\hat{x_n} \xrightarrow{w*} \hat{x}$. Since M'' is weak* continuous,

$$\widehat{Mx_n} = M''\hat{x_n} \xrightarrow{w*} M''\hat{x} = \widehat{Mx}$$

in U''.

Since U is reflexive, weak* convergence is the same as weak convergence, so $Mx_n \xrightarrow{w} Mx$ as required.

2.2 Exercise 3

Denote by I the identity map $X \to X$. Consider the transpose of I, $I' : X' \to X'$.

Let $l \in X', x \in X$.

$$(I'l)x = l(Ix) = l(x).$$

Therefore $I'l = l$.

2.3 Exercise 4

(i)

Let X, Y be complex Hilbert spaces. Let $M : X \to Y$ be a bounded linear transformation. Let $M^* : Y \to X$ be the adjoint of M, such that $\langle Mx, y\rangle = \langle x, M^*y\rangle$ for all $x \in X$, $y \in Y$.

Let $x \in X$ such that $\|x\| \leq 1$. Then

$$\begin{aligned}
\|Mx\|^2 &= \langle Mx, Mx \rangle_Y \\
&= \langle x, M^*Mx \rangle_X \\
&\leq \|x\| \|M^*Mx\| \quad \text{(Cauchy-Schwarz Inequality)} \\
&\leq \|x\| \|M^*M\| \|x\| \\
&\leq \|M^*M\| \|x\| \quad \text{(since } \|x\| \leq 1\text{)} \\
&\leq \|M^*M\| \\
&= \sup_{\|x\|=1} \|M^*Mx\| \\
&\leq \sup_{\|x\|=1} \|M^*\| \|Mx\| \\
&= \|M^*\| \|M\|
\end{aligned}$$

Thus $\|M\|^2 = \sup_{\|x\| \leq 1} \|Mx\|^2 \leq \|M^*\| \|M\|$. Therefore $\|M\| \leq \|M^*\|$.

In Hilbert space, $M^{**} = M$ since $\langle Mx, y \rangle = \langle x, M^*y \rangle = \langle M^{**}x, y \rangle$ for all $x \in X$, $y \in Y$. Thus, replacing M by M^* in the above working yields $\|M^*\| \leq \|M\|$.

Therefore $\|M\| = \|M^*\|$.

(ii)

Let $y \in N_{M^*}$. $M^*y = 0$. Hence $\langle x, M^*y \rangle = 0 = \langle Mx, y \rangle$ for all $x \in X$. This means $y \in R_M^\perp$.

Let $y \in R_M^\perp$. $\langle Mx, y \rangle = 0 = \langle x, M^*y \rangle$ for all $x \in X$. This means $M^*y = 0$, thus $y \in N_{M^*}$.

We have proved $N_{M^*} = R_M^\perp$.

(iii)

Replace M by M^* in the above, and using the fact that $M^{**} = M$, we get $N_M = R_{M^*}^\perp$.

(iv)

$$\begin{aligned}\langle x,(M+N)^*y\rangle &= \langle (M+N)x,y\rangle\\ &= \langle Mx+Nx,y\rangle\\ &= \langle Mx,y\rangle+\langle Nx,y\rangle\\ &= \langle x,M^*y\rangle+\langle x,N^*y\rangle\\ &= \langle x,(M^*+N^*)y\rangle\end{aligned}$$

for all $x \in X$, $y \in Y$. Thus, $\langle x,(M+N)^*y-(M^*+N^*)y\rangle \equiv 0$ which implies that $(M+N)^*y-(M^*+N^*)y \equiv 0$ thus $(M+N)^* = M^*+N^*$.

2.4 Exercise 6

Assume M_n converges to M weakly. Then $\lim_{n\to\infty}(M_nx,l) = (Mx,l)$ for all $l \in U'$ and all $x \in X$.

Thus $\lim_{n\to\infty}(x,M_n'l) = (x,M'l)$ for all $l \in U'$, $x \in$X.

Let $\phi \in X''$. Since X is reflexive, $\phi = \hat{x}$ for some $\hat{x} \in X''$. Then

$$\begin{aligned}\lim_{n\to\infty}\hat{x}(M_n'l) &= \lim_{n\to\infty}M_n'l(x)\\ &= \lim_{n\to\infty}(x,M_n'l)\\ &= (x,M'l)\\ &= M'l(x)\\ &= \hat{x}(M'l).\end{aligned}$$

Therefore M_n' converges to M' weakly.

2.5 Exercise 7

(a)

Let D be dense in X such that $s-\lim M_nx$ exists. Then for $d \in D$, $\{M_nd\}$ is Cauchy. There exists $N \in \mathbb{N}$ such that for all $n,m \geq N$, $\|M_nd-M_md\| < \epsilon$.

Let $x \in X$. There exists $d \in D$ such that $\|x-d\| < \epsilon$. For $n,m \geq N$,

$$\begin{aligned}\|M_nx-M_mx\| &= \|M_n(x+d-d)-M_m(x+d-d)\|\\ &= \|M_nd+M_n(x-d)-M_md-M_m(x-d)\|\\ &\leq \|M_nd-M_md\|+\|M_n(x-d)\|+\|M_m(x-d)\|\\ &< \epsilon+\|M_n\|\|x-d\|+\|M_m\|\|x-d\|\\ &< \epsilon+c\epsilon+c\epsilon.\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $\{M_n x\}$ is Cauchy thus converges since U is a Banach space, i.e. there exists a limit M such that $\lim_{n\to\infty} \|M_n x - Mx\| = 0$, for all x.

(b)

Theorem 2.4. Let X, U be Banach spaces, M_n a sequence of linear maps: $X \to U$, uniformly bounded in norm:

$$\|M_n\| \leq c \quad \text{for all } n.$$

Suppose further that $w - \lim M_n x$ exists for a dense set of x in X. Then $\{M_n\}$ converges weakly, i.e. the w-limit exists for all x in X.

Proof. Let D be the dense set in X such that $w - \lim M_n x$ exists, i.e.

$$\lim_{n\to\infty} |l(M_n d) - l(Md)| = 0$$

for all $d \in D$, $l \in U'$. Then, $\{l(M_n d)\}$ is Cauchy, so there exists $N \in \mathbb{N}$ such that for $n, m \geq N$,

$$|l(M_n d) - l(M_m d)| < \epsilon.$$

Let $x \in X$. There exists $d \in D$ such that $\|x - d\| < \epsilon$. For $n, m \geq N$,

$$\begin{aligned}
|l(M_n x) - l(M_m x)| &= |l(M_n d) + l(M_n(x-d)) - l(M_m d) - l(M_m(x-d))| \\
&\leq |l(M_n d) - l(M_m d)| + |l(M_n(x-d))| + |l(M_m(x-d))| \\
&< \epsilon + \|l\|\|M_n\|\|x-d\| + \|l\|\|M_m\|\|x-d\| \\
&\leq \epsilon + \|l\|c\epsilon + \|l\|c\epsilon
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary and $\|l\| < \infty$, $\{l(M_n x)\}$ is Cauchy in $\mathbb{R}$ or $\mathbb{C}$ and thus converges. Thus $\{M_n\}$ converges weakly. □

Exercise 11

Assume that the range R_M is a finite-codimensional subspace of U, i.e. $\text{codim}(R_M) = \dim(U/R_M) < \infty$.

$\ker M = M^{-1}\{0\}$ is a closed subspace, thus $X/\ker M$ is a Banach space.

$$\overline{M} : X/\ker M \to U$$

defined by

$$\overline{M}(x + \ker M) = M(x)$$

is an injective bounded linear operator. We note that $R_M = M(X) = \overline{M}(X/\ker M)$.

Choose a basis $f_1, \ldots, f_n$ of U/R_M and consider the operator

$$S : X/\ker M \oplus \mathbb{R}^n \to U = R_M \oplus U/R_M$$
$$S(\bar{x}, r) = \overline{M}(\bar{x}) + \sum_{i=1}^{n} r_i f_i.$$

We observe that S is a continuous bijection, hence a homeomorphism by the open mapping theorem. $R_M = S(X/\ker M \oplus 0)$ is the image of a closed subspace of $X/\ker M$, thus R_M is closed since S maps closed sets onto closed sets.