Lax Solution Part 2

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Textbook: Functional Analysis by Peter D. Lax
Exercises: Ch 8: Q6,7. Ch 10: Q5,6.

1 Chapter 8

1.1 Exercise 6

($\subseteq$) Let C be the convex hull of M. The closure of the convex hull of M, denoted $cl(C)$, is the smallest closed set containing C. Chapter 1 Theorem 6 says that C is the smallest convex set containing M. Thus $cl(C)$ is a closed set containing M.

Let $x, y \in cl(C)$, $0 \leq a \leq 1$. There exists sequences (x_n), (y_n) in C converging to x, y respectively. Then $(ax_n + (1-a)y_n)$ is a sequence in C converging to $ax + (1-a)y \in cl(C)$. Thus $cl(C)$ is a closed **convex** set containing M. Therefore

$$\breve{M} \subseteq cl(C)$$

since $\breve{M}$ is the intersection of all closed convex sets containing M.

($\supseteq$) Let $x \in cl(C)$. There exists a sequence (x_n) in C converging to x. Let D be any closed convex set containing M. Since C is the smallest convex set containing M, so $C \subseteq D$. So (x_n) is a sequence in D. Since D is closed, so $x \in D$. Therefore $x \in \breve{M}$, the intersection of all closed convex sets containing M. Thus

$$cl(C) \subseteq \breve{M}.$$

1.2 Exercise 7

(i)

$$\begin{aligned} S_M(l+m) &= \sup_{y\in M}[l(y)+m(y)] \\ &\le \sup_{y\in M} l(y) + \sup_{y\in M} m(y) \\ &= S_M(l) + S_M(m) \end{aligned}$$

(ii) $S_M(0) = \sup_{y\in M} 0 = 0$
(iii) For $a > 0$,

$$S_M(al) = \sup_{y\in M} al(y) = a \sup_{y\in M} l(y) = aS_M(l)$$

(iv) Let $M \subseteq N$.

$$S_M(l) = \sup_{y\in M} l(y) \le \sup_{y\in N} l(y) = S_N(l)$$

(v)

$$\begin{aligned} S_{M+N}(l) &= \sup_{y\in M+N} l(y) \\ &= \sup_{m\in M, n\in N} l(m+n) \\ &= \sup_{m\in M, n\in N} (l(m)+l(n)) \\ &= \sup_{m\in M} l(m) + \sup_{n\in N} l(n) \\ &= S_M(l) + S_N(l) \end{aligned}$$

(vi)

$$\begin{aligned} S_{-M}(l) &= \sup_{y\in -M} l(y) \\ &= \sup_{z\in M} l(-z) \\ &= \sup_{z\in M} -l(z) \\ &= S_M(-l) \end{aligned}$$

(vii) Since $M \subseteq \overline{M}$, by monotonicity, $S_M(l) \leq S_{\overline{M}}(l)$.

$$\begin{aligned}
S_{\overline{M}}(l) &= \sup_{y \in \overline{M}} l(y) \\
&= \sup_{\substack{\text{convergent} \\ (y_n) \text{ in } M}} l(\lim_{n\to\infty} y_n) \\
&= \sup_{\substack{\text{convergent} \\ (y_n) \text{ in } M}} \lim_{n\to\infty} l(y_n) \quad \text{(by continuity of } l) \\
&= \sup A, \text{ where } A = \{\lim_{n\to\infty} l(y_n) \mid (y_n) \text{ a convergent sequence in } M\}
\end{aligned}$$

Let $\lim_{n\to\infty} l(y_n) \in A$. We have $\sup_{y\in M} l(y) \geq l(y_n)$ for all n, so $\sup_{y \in M} l(y) \geq \lim_{n\to\infty} l(y_n)$, i.e. $S_M(l) = \sup_{y\in M} l(y)$ is an upper bound of A. Then $S_{\overline{M}}(l) = \sup A \leq S_M(l)$.

(viii) Denote the convex hull of M by C.

$$\begin{aligned}
S_C(l) &= \sup_{y\in C} l(y) \\
&= \sup\{l(\sum_{j=1}^n a_j x_j) \mid x_j \in M, a_j \geq 0, \sum_{j=1}^n a_j = 1\} \text{ (since each } y \in C \\
&\quad \text{is a convex combination of points in } M) \\
&= \sup\{\sum_{j=1}^n a_j l(x_j) \mid x_j \in M, a_j \geq 0, \sum_{j=1}^n a_j = 1\} \text{ (by linearity of } l) \\
&= \sum_{j=1}^n a_j \sup\{l(x) \mid x \in M\}, \text{ where } a_j \geq 0, \sum_{j=1}^n a_j = 1 \\
&= \sup\{l(x) \mid x \in M\} \\
&= S_M(l)
\end{aligned}$$

2 Chapter 10

2.1 Exercise 5

Let C be a weakly sequentially compact set. Suppose to the contrary C is not bounded, i.e. for all $x \in X$, for all $r > 0$, there exists $y \in C$ such that $\|x - y\| \geq r$. Thus, we may construct a sequence (y_n) in C such that

$$\|y_n\| \geq n$$

for all n (by choosing $x = 0$, $r = n$).

(y_n) has a subsequence (y_{n_k}) that is weakly convergent, with

$$\|y_{n_k}\| \geq n_k \geq k$$

for all $k \in \mathbb{N}$. This contradicts Theorem 4' which states that a weakly convergent sequence in a normed linear space is uniformly bounded in norm.

2.2 Question 6

Let U be a Banach space that is the dual of another Banach space X. Assume that the sequence $\{u_n\}$ in U is weak* convergent to u, i.e. $\lim u_n(x) = u(x)$ for all x in X.

We have $\|u_n(x)\| \leq \|u_n\|\|x\|$ for all x in X. Thus,

$$\liminf \|u_n(x)\| \leq \liminf \|u_n\|\|x\|.$$

Lemma 2.1. $\liminf \|u_n(x)\| = \|u(x)\|$.

Proof. By continuity of the norm, we have $\lim \|u_n(x)\| = \|\lim u_n(x)\| = \|u(x)\|$. Hence the sequence $(\|u_n(x)\|)$ converges and so $\liminf \|u_n(x)\| = \lim \|u_n(x)\| = \|u(x)\|$. □

$$\|u(x)\| \leq \liminf \|u_n\|\|x\|$$
$$\sup_{\|x\|=1} \|u(x)\| \leq \liminf \|u_n\|$$

Thus $\|u\| \leq \liminf \|u_n\|$.