Lax Functional Analysis Solutions

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February 5, 2016

Textbook: Functional Analysis by Peter D. Lax
Exercises: Ch 1: 6. Chp 3: 1. Ch5: 3,4. Ch6: 1,3,6,8,9.

1 Chapter 1

1.1 Exercise 6

(i) By definition,

$$\hat{S} = \bigcap_{i \in I} K_i$$

, where K_i is a convex set containing S. Theorem 5(vi) states that the intersection of an arbitrary collection of convex sets is convex. Thus $\hat{S}$ is convex.

Any convex set containing S must be one of the K_i, and $\hat{S} = \bigcap_{i \in I} K_i \subseteq K_i$. Thus $\hat{S}$ is the smallest convex set containing S.

(ii) Let C be the set of all convex combinations of points of S. Let $x = \sum_{j=1}^n a_j x_j$ be a convex combination of points of S. ($a_j \geq 0$, $\sum_{j=1}^n a_j = 1$, $x_j \in S$)

Since $x_j \in S \subseteq K_i$ for all i, we have that $x_j \in \bigcap_{i \in I} K_i = \hat{S}$ for all $1 \leq j \leq n$.

We quote Theorem 4:

> "Let K be a convex subset of a linear space X over the reals. Suppose that $x_1, \ldots, x_n$ belong to K; then so does every x of the form $x = \sum_{j=1}^n a_j x_j$, $a_j \geq 0$, $\sum_{j=1}^n a_j = 1$."

Applying Theorem 4 to $\hat{S}$ (which we proved to be convex), $\hat{S}$ contains all convex combinations of points of S. We have shown $C \subseteq \hat{S}$.

To show $\hat{S} \subseteq C$, it suffices by (i) to show that C is a convex set containing S. We show C is convex: Let $x = \sum_{j=1}^{n} a_j x_j$, $y = \sum_{i=1}^{m} b_i y_i$ be two convex combinations of points in S. For $0 \leq t \leq 1$,

$$tx + (1-t)y = \sum_{j=1}^{n} a_j t x_j + \sum_{i=1}^{m} (1-t) b_i y_i$$

is also a convex combination of points in S since $\sum_{j=1}^{n} a_j t + \sum_{i=1}^{m} (1-t) b_i = 1$ and $a_j t, (1-t)b_i \geq 0$.

Let $s \in S$. s can be written as the trivial convex combination $s = 1s$, thus s is in C. Thus C contains S. Done.

2 Chapter 3

2.1 Exercise 1

Assume $p_K(x) < 1$. Suppose to the contrary x is not an interior point of K, i.e. there exists $y \in X$ such that for all $\epsilon > 0$,

$$x + ty \notin K$$

for some $|t| < \epsilon$. This means $p_K(x + ty) > 1$.

By subadditivity of p_K,

$$1 < p_K(x + ty) \leq p_K(x) + p_K(ty)$$

Case 1) $t = 0$. Contradiction obtained since $p_K(0) = 0$ implies $p_K(x) > 1$.

Case 2) $t > 0$. By positive homogeneity of p_K,

$$1 < p_K(x + ty) \leq p_K(x) + t p_K(y)$$

As $\epsilon \to 0$, $t \to 0$ so $p_K(x) \geq 1$, a contradiction.

Case 3) $t < 0$.

$$1 < p_K(x + ty) \leq p_K(x) - t p_K(-y)$$

Similarly as $\epsilon \to 0$, $t \to 0$ so $p_K(x) \geq 1$.

Conversely, assume x is an interior point of K. Choose $y = x \in X$. There exists $\epsilon > 0$ such that

$$x + ty = (1+t)x \in K$$

for all real t, $|t| < \epsilon$. We choose $t > 0$. $p_K((1+t)x) \leq 1$ since $(1+t)x \in K$

By positive homogeneity, $(1+t)p_K(x) \leq 1$ which implies

$$p_K(x) \leq \frac{1}{1+t} < 1$$

3 Chapter 5

3.1 Exercise 3

We will quote and use a well-known basic theorem:

Theorem 3.1. Let X be a normed space. Then X is complete iff the series $\sum_{n=1}^{\infty} x_n$ converges, where (x_n) is any sequence in X such that $\sum_{n=1}^{\infty} \|x_n\| < \infty$.

Let $(x_n + Y)$ be a sequence in X/Y such that $\sum_{n=1}^{\infty} \|x_n + Y\| < \infty$. Recall that $\|x_n + Y\| = \inf_{y \in Y} \|x_n + y\|$. By definition of infimum, there exists $y_n \in Y$ such that

$$\|x_n + y_n\| < \|x_n + Y\| + \frac{1}{2^n}$$

Thus

$$\sum_{n=1}^{\infty} \|x_n + y_n\| < \sum_{n=1}^{\infty} \|x_n + Y\| + \sum_{n=1}^{\infty} \frac{1}{2^n} < \infty$$

By Theorem 3.1, since X is a Banach space, $\sum_{n=1}^{\infty}(x_n + y_n)$ converges to some $l \in X$.

We shall show that $\sum_{n=1}^{\infty}(x_n + Y)$ converges to $l + Y \in X/Y$.

$$\begin{aligned}
\|\sum_{n=1}^{k}(x_n + Y) - (l + Y)\| &= \|\sum_{n=1}^{k} x_n - l + Y\| \\
&= \inf_{y \in Y} \|\sum_{n=1}^{k} x_n - l + y\| \\
&\leq \|\sum_{n=1}^{k} x_n - l + \sum_{n=1}^{k} y_n\| \\
&= \|\sum_{n=1}^{k}(x_n + y_n) - l\| \\
&\to 0 \text{ as } k \to \infty
\end{aligned}$$

Thus $\sum_{n=1}^{\infty}(x_n + Y) \to l + Y$. We are done, by Theorem 3.1.

3.2 Exercise 4

First we prove the hint, i.e. we assume the fact that all norms are equivalent on finite-dimensional spaces to show that every finite-dimensional subspace is complete.

Proof. Let X be a finite n-dimensional subspace with basis $\{e_1, e_2, \ldots, e_n\}$, equipped with a norm $\|\cdot\|$. Let (x_k) be a Cauchy sequence in X. We have that $\|\cdot\|$ is equivalent to the l^1-norm $\|\cdot\|_1$, i.e. there exists $0 < c_1 \leq c_2$ such that $c_1\|x\|_1 \leq \|x\| \leq c_2\|x\|_1$ for all $x \in X$. We write each $x_k = \sum_{i=1}^n \alpha_{k_i} e_i$, where $\alpha_{k_i} \in \mathbb{R}$. (Proof also works for $\mathbb{C}$)

There exists $N \in \mathbb{N}$ such that for $a, b \geq N$,

$$\begin{aligned} \epsilon &> \|x_a - x_b| \\ &\geq c_1\|x_a - x_b\|_1 \\ &= c_1 \sum_{i=1}^n |\alpha_{a_i} - \alpha_{b_i}| \\ &\geq |\alpha_{a_i} - \alpha_{b_i}| \end{aligned}$$

for each $1 \leq i \leq n$.

Thus (α_{k_i}) is a Cauchy sequence in $\mathbb{R}$ for each i. $\mathbb{R}$ is complete, thus $\lim_{k\to\infty} \alpha_{k_i} := \beta_i$ is in $\mathbb{R}$ for each i. We define $x := (\beta_1, \ldots, \beta_n) = \sum_{i=1}^n \beta_i e_i \in X$.

$$\begin{aligned} \|x_k - x\| &= \|\sum_{i=1}^n (\alpha_{k_i} - \beta_i)e_i\| \\ &\leq c_2\|\sum_{i=1}^n (\alpha_{k_i} - \beta_i)e_i\|_1 \\ &= c_2 \sum_{i=1}^n |\alpha_{k_i} - \beta_i| \\ &\to 0 \text{ as } k \to \infty \end{aligned}$$

Therefore, $x_k \to x$ in X, i.e. X is complete. Hint proved. □

Next, we use the fact that the closure of X, $\overline{X}$, is the set of all limits of all convergent sequences of points in X. Let $x \in \overline{X}$. There exists a convergent (hence Cauchy) sequence of points (x_n) in X which converges to x. Since X is complete, $x \in X$. Thus $\overline{X} \subseteq X$. $X \subseteq \overline{X}$ is clear, thus X is closed.

4 Chapter 6

4.1 Exercise 1

Let $\|\cdot\|$ be a norm that satisfies the parallelogram identity:

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

for all x, y in X. Define

$$\langle x, y\rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2)$$

for all $x, y \in X$. We are proving the real case first, where the field of scalars is $\mathbb{R}$.

We need to prove that $\langle x, y\rangle$ is indeed a scalar product. Symmetry, $\langle x, y\rangle = \langle y, x\rangle$ is clear.

Positivity property holds: $\langle x, x\rangle = \frac{1}{4}(\|x+x\|^2) = \|x\|^2 \geq 0$ with equality only when $x = \mathbf{0}$.

Lemma 4.1. Additive in first component: $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$

By the parallelogram law,

$$2\|x+z\|^2 + 2\|y\|^2 = \|x+y+z\|^2 + \|x-y+z\|^2$$

Rearranging, we get

$$\|x+y+z\|^2 = 2\|x+z\|^2 + 2\|y\|^2 - \|x-y+z\|^2$$

Since x, y are arbitrary, setting $x = y$, $y = x$ gives

$$\|y+x+z\|^2 = 2\|y+z\|^2 + 2\|x\|^2 - \|y-x+z\|^2$$

Therefore

$$\begin{aligned}\|x+y+z\|^2 &= \frac{1}{2}(2\|x+z\|^2 + 2\|y\|^2 - \|x-y+z\|^2) \\ &+ \frac{1}{2}(2\|y+z\|^2 + 2\|x\|^2 - \|y-x+z\|^2) \\ &= \|x\|^2 + \|y\|^2 + \|x+z\|^2 + \|y+z\|^2 - \frac{1}{2}\|x-y+z\|^2 - \frac{1}{2}\|y-x+z\|^2\end{aligned}$$

Replacing z by $-z$ yields

$$\|x+y-z\|^2 = \|x\|^2 + \|y\|^2 + \|x-z\|^2 + \|y-z\|^2 - \frac{1}{2}\|x-y-z\|^2 - \frac{1}{2}\|y-x-z\|^2$$

Thus,

$$\begin{aligned}\langle x+y,z\rangle &= \frac{1}{4}(\|x+y+z\|^2 - \|x+y-z\|^2)\\ &= \frac{1}{4}(\|x+z\|^2 - \|x-z\|^2) + \frac{1}{4}(\|y+z\|^2 - \|y-z\|^2)\\ &= \langle x,z\rangle + \langle y,z\rangle\end{aligned}$$

Lemma 4.1 proved.

Lemma 4.2. $\langle \alpha x, y\rangle = \alpha\langle x,y\rangle$ for all $\alpha \in \mathbb{R}$

Clearly true for $\alpha = -1$. By Lemma 4.1 (addivity in first component) and induction we have $\langle \alpha x, y\rangle = \alpha\langle x,y\rangle$ for all $\alpha \in \mathbb{Z}$. If $\alpha = \frac{p}{q}$ with $p, q \in \mathbb{Z}$, $q \neq 0$ we have

$$q\langle \alpha x, y\rangle = q\langle \frac{p}{q}x, y\rangle = p\langle x,y\rangle$$

Thus dividing by q yields $\langle x,y\rangle = \alpha\langle x,y\rangle$ for all $\alpha \in \mathbb{Q}$.

We note that the function $(x,y) \mapsto \langle x,y\rangle$ is continuous with respect to the norm $\|\cdot\|$ since the norm itself is continuous. Thus, we can define a continuous function

$$f(t) = \frac{1}{t}\langle tx, y\rangle$$

which is equal to $\langle x,y\rangle$ for all $t \in \mathbb{Q}\setminus\{0\}$. By using the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$ and taking limits, $f(t) = \frac{1}{t}\langle tx,y\rangle = \langle x,y\rangle$ for all $t \in \mathbb{R}\setminus\{0\}$. Thus $\langle tx, y\rangle = t\langle x,y\rangle$ for all $t \in \mathbb{R}$. (The case $t=0$ is trivial).

Finally, we deal with the complex case. We will only focus on those parts that are different from the real case, namely sesquilinearity and skew symmetry. Define

$$\langle x,y\rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2) + \frac{i}{4}(\|x+iy\|^2 - \|x-iy\|^2)$$

Observe that

$$\begin{aligned}\langle ix,y\rangle &= \frac{1}{4}(\|ix+y\|^2 - \|ix-y\|^2) + \frac{i}{4}(\|ix+iy\|^2 - \|ix-iy\|^2)\\ &= i\langle x,y\rangle\end{aligned}$$

Similar to the real case, this can be extended to $\langle \alpha x, y\rangle = \alpha\langle x,y\rangle$ for all $\alpha \in \mathbb{C}$.

$$\begin{aligned}\overline{\langle y,x\rangle} &= \frac{1}{4}(\|y+x\|^2 - \|y-x\|^2) - \frac{i}{4}(\|y+ix\|^2 - \|y-ix\|^2)\\ &= \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2) - \frac{i}{4}(\|-iy+x\|^2 - \|iy+x\|^2)\\ &= \langle x,y\rangle\end{aligned}$$

Thus skew symmetry is true.

Sesquilinearity is true since

$$\begin{aligned}\langle x, \alpha y\rangle &= \overline{\langle \alpha y, x\rangle} \\ &= \overline{\alpha\langle y, x\rangle} \\ &= \overline{\alpha}\,\overline{\langle y, x\rangle} \\ &= \overline{\alpha}\langle x, y\rangle\end{aligned}$$

We can do a routine check that

$$\begin{aligned}&\frac{1}{4}(\|x+y\|^2 - \|x-y\|^2) + \frac{i}{4}(\|x+iy\|^2 - \|x-iy\|^2) \\ &= \mathrm{Re}\langle x, y\rangle + i\mathrm{Im}\langle x, y\rangle\end{aligned}$$

by using $\|x+y\|^2 = \langle x+y, x+y\rangle$, etc., thus proving that our definition is consistent with the definition of a norm, $\|x\| = \sqrt{\langle x, x\rangle}$.

4.2 Exercise 3

Let (x_n) be a Cauchy sequence in l^2, where $x_n = (a_{n_1}, a_{n_2}, \dots)$, $\sum |a_{n_j}|^2 < \infty$, $a_{n_j} \in \mathbb{R}$ or $\mathbb{C}$. There exists $N \in \mathbb{N}$ such that for $n, m \geq N$,

$$\|x_n - x_m\|_2^2 = \sum_{j=1}^{\infty} |a_{n_j} - a_{m_j}|^2 < \epsilon^2$$

Therefore, each $|a_{n_j} - a_{m_j}|^2 < \epsilon^2$ for all j. This implies

$$|a_{n_j} - a_{m_j}| < \epsilon$$

for all j.

This implies that for each fixed j, (a_{n_j}) is a Cauchy sequence in $\mathbb{R}$ (or $\mathbb{C}$), and thus converges to $b_j \in \mathbb{R}$ (or $\mathbb{C}$). Define

$$x := (b_1, b_2, \dots)$$

For any $k \in \mathbb{N}$ and $n, m \geq N$,

$$\sum_{j=1}^{k} |a_{n_j} - a_{m_j}|^2 \leq \sum_{j=1}^{\infty} |a_{n_j} - a_{m_j}|^2 < \epsilon^2$$

Taking limits as $m \to \infty$,

$$\sum_{j=1}^{k} |a_{n_j} - b_j|^2 \leq \epsilon^2 \tag{4.1}$$

for any $k \in \mathbb{N}$, $n \geq N$.

By Minkowski's inequality,

$$\begin{aligned}(\sum_{j=1}^{k} |b_j|^2)^{\frac{1}{2}} &\leq (\sum_{j=1}^{k} |a_{N_j} - b_j|^2)^{\frac{1}{2}} + (\sum_{j=1}^{k} |a_{N_j}|^2)^{\frac{1}{2}} \\ &\leq \epsilon + (\sum_{j=1}^{\infty} |a_{N_j}|^2)^{\frac{1}{2}} \\ &= \epsilon + \|x_N\|^2\end{aligned}$$

Letting $k \to \infty$, $(\sum_{j=1}^{\infty} |b_j|^2)^{\frac{1}{2}} < \infty$ so $x \in l^2$. Finally, letting $k \to \infty$ in *Equation* 4.1 gives

$$\sum_{j=1}^{\infty} |a_{n_j} - b_j|^2 \leq \epsilon^2$$

for all $n \geq N$.

Thus,

$$\lim_{n\to\infty} \|x_n - x\|_2 = \lim_{n\to\infty} (\sum_{j=1}^{\infty} |a_{n_j} - b_j|^2)^{\frac{1}{2}} \to 0$$

This shows that l^2 is complete.

4.3 Exercise 6

Let C denote the closed linear span of $\{x_j\}$, i.e. $C = cl(span(\{x_k\}))$. Let S denote all vectors of the form $x = \sum a_j x_j$ where the a_j are complex numbers such that $\sum |a_j|^2 < \infty$.

We want to prove $C = S$.

Let $x \in S$, $x = \sum a_j x_j$, where $\sum |a_j|^2 < \infty$. First we need to ensure that $x = \sum a_j x_j$ converges.

Remark 4.3. $\sum |a_j|^2 < \infty$ implies that at most countable many a_j are nonzero.

Let $y_k := \sum_{j=1}^k a_j x_j$. For $n > m$,

$$\|y_n - y_m\|^2 = \| \sum_{j=m+1}^{n} a_j x_j \|^2 = \sum_{j=m+1}^{n} |a_j|^2 \to 0$$

as $n, m \to \infty$ because $\sum |a_j|^2$ converges. Thus (y_k) is a Cauchy sequence in H and thus converges (to x).

We quote and use:

> Theorem 7: The point y of a Hilbert space H belongs to the closed linear span Y of the set $\{y_j\}$ iff every vector z that is orthogonal to all y_j is orthogonal also to y: $\langle y, z\rangle = 0$ for all z that satisfy $\langle y_j, z\rangle = 0$ for all j.

Let $z \in H$ such that $\langle x_j, z\rangle = 0$ for all j.

$$\begin{aligned}\langle x, z\rangle &= \langle \sum a_j x_j, z\rangle \\ &= \lim_{k\to\infty} \langle \sum_{j=1}^{k} a_j x_j, z\rangle \\ &= \lim_{k\to\infty} (\sum_{j=1}^{k} a_j \langle x_j, z\rangle) \\ &= 0\end{aligned}$$

Therefore $x \in C$. Thus $S \subseteq C$.

Let $x \in C$. Consider $y = x - \sum \langle x, x_j\rangle x_j$. (Remark: In the next question Exercise 8 (†), we will prove that $\{x_j \mid \langle x, x_j\rangle \neq 0\}$ is at most countable.)

$$\begin{aligned}\langle y, x_k\rangle &= \langle x - \sum \langle x, x_j\rangle x_j, x_k\rangle \\ &= \langle x, x_k\rangle - \lim_{n\to\infty} \langle \sum_{j=1}^{n} \langle x, x_j\rangle x_j, x_k\rangle \\ &= \langle x, x_k\rangle - \langle x, x_k\rangle \\ &= 0\end{aligned}$$

By Theorem 7, $\langle y, y\rangle = 0$ which implies $y = 0$. Thus $x = \sum a_j x_j$ where $a_j = \langle x, x_j\rangle$. By Bessel's inequality, $\sum |a_j|^2 \leq \|x\|^2 < \infty$. Therefore, $C \subseteq S$.

Finally,

$$\begin{aligned}
\|x\|^2 &= \langle \sum a_j x_j, \sum a_j x_j \rangle \\
&= \lim_{k\to\infty} \langle \sum_{j=1}^{k} a_j x_j, \sum_{j=1}^{k} a_j x_j \rangle \\
&= \lim_{k\to\infty} \sum_{j=1}^{k} |a_j|^2 \\
&= \sum_{j=1}^{\infty} |a_j|^2
\end{aligned}$$

We have proved all the parts of lemma 8.

4.4 Exercise 8

Let $\{x_j\}$ and $\{y_j\}$ be two orthonormal bases in H.

Case 1) At least one of the bases is finite.

WLOG, we assume $\{x_j\}$ is finite, so $\{x_j\} = \{x_1, \ldots, x_n\}$. By definition of orthonormal basis, $cl(span(\{x_1, \ldots, x_n\})) = H.$. Since every finite dimensional subspace of a normed linear space is closed (Homework Ex. 4), $span(\{x_1, \ldots, x_n\}) = H$.

Orthonormality implies linear independence: If $c_1x_1 + c_2x_2 + \cdots + c_nx_n = 0$, consider $\langle c_1x_1 + \cdots + c_nx_n, x_j \rangle = c_j = 0$, for all $1 \le j \le n$. This means $\{x_1, \ldots, x_n\}$ is linearly independent. Thus by Linear Algebra, $\{x_1, \ldots, x_n\}$ is a basis for H, and the (algebraic) dimension of H is n. Since $\{y_j\}$ is also linearly independent, $|\{y_j\}| \le n = |\{x_j\}|$. In particular, $\{y_j\}$ is finite. By exchanging the role of $\{x_j\}$ and $\{y_j\}$ in the argument above, we have $|\{x_j\}| \le |\{y_j\}|$.

Therefore, $|\{x_j\}| = |\{y_j\}|$.

Case 2) Both bases are infinite. We write $A := \{x_j\}$, $B := \{y_j\}$. We will prove that

$$A = \bigcup_{y_k \in B} \{x_j \in A \mid \langle y_k, x_j \rangle \neq 0\}$$

$\bigcup_{y_k \in B} \{x_j \in A \mid \langle y_k, x_j \rangle \neq 0\} \subseteq A$ is clear.

Let $x_j \in A$. Suppose to the contrary $x_j \notin \bigcup_{y_k \in B} \{x_j \in A \mid \langle y_k, x_j \rangle \neq 0\}$. Then $\langle y_k, x_j \rangle = 0$ for all $y_k \in B$. By Theorem 7, since $x_j \in cl(span(\{y_j\})) = H$, we must have $\langle x_j, x_j \rangle = 0$ which implies $x_j = 0$. This is a clear contradiction since $\|x_j\| = 1$. So A is indeed the union above.

Next, we show that each set $\{x_j \in A \mid \langle y_k, x_j\rangle \neq 0\} := S_k$ is countable (for fixed y_k) $(\dagger)$[1]. Note that $S_k = \bigcup_{n=1}^{\infty}\{x_j \in A \mid |\langle y_k, x_j\rangle| > \frac{1}{n}\}$.

Let $a_1, \dots, a_m$ be m elements in $\{x_j \in A \mid |\langle y_k, x_j\rangle| > \frac{1}{n}\}$. By Bessel's inequality,

$$m \cdot (\frac{1}{n})^2 \leq \sum_{j=1}^{m} |\langle y_k, x_j\rangle|^2 \leq \|y_k\|^2$$

So $m \leq \|y_k\|^2/(\frac{1}{n})^2$, which is finite. Thus S_k is countable.

Thus, $\bigcup_{y_k\in B}\{x_j \in A \mid \langle y_k, x_j\rangle \neq 0\} = \bigcup S_k$ has cardinality less than $\aleph_0 \times |B|$. Since B is infinite, $\aleph_0 \leq |B|$. Thus $|A| \leq \aleph_0 \times |B| \leq |B|^2 = |B|$. By symmetry (repeat argument with A, B interchanged), $|B| \leq |A|$. So $|A| = |B|$.

4.5 Exercise 9

Define $M : H \to H$, $x = \sum a_j x_j \mapsto y = \sum a_j y_j$. Let $z_1 = \sum\langle z_1, x_j\rangle x_j$, $z_2 = \sum\langle z_2, x_j\rangle x_j$.

$$\begin{aligned}\|M(z_1) - M(z_2)\|^2 &= \|\sum\langle z_1, x_j\rangle y_j - \sum\langle z_2, x_j\rangle y_j\|^2\\ &= \|\sum(\langle z_1, x_j\rangle - \langle z_2, x_j\rangle)y_j\|^2\\ &= \|\sum\langle z_1 - z_2, x_j\rangle y_j\|^2\\ &= \langle\sum\langle z_1 - z_2, x_j\rangle y_j, \sum\langle z_1 - z_2, x_j\rangle y_j\rangle\\ &= \sum|\langle z_1 - z_2, x_j\rangle|^2\end{aligned}$$

$$\begin{aligned}\|z_1 - z_2\|^2 &= \|\sum\langle z_1 - z_2, x_j\rangle x_j\|^2\\ &= \sum|\langle z_1 - z_2, x_j\rangle|^2\end{aligned}$$

Therefore $\|M(z_1) - M(z_2)\| = \|z_1 - z_2\|$ so M is an isometry.

M is onto: Let $y = \sum\langle y, y_j\rangle y_j \in H$. Consider $\sum\langle y, y_j\rangle x_j \in H$. Then $M(\sum\langle y, y_j\rangle x_j) = y$.

$M(0) = M(\sum 0x_j) = \sum 0y_j = 0$.

Let $\phi : H \to H$ be a **linear** isometry of H onto H mapping 0 to 0. Remark: We believe the question needs to have an extra condition "linear", i.e. it should be *"every linear isometry of H onto H mapping $0 \to 0$ can be*

[1] This argument is also applicable for the previous question Exercise 6.

obtained in this fashion", since any isometry obtained in the given fashion is necessarily linear.

Let $z_1, z_2 \in H$. We show that ϕ preserves inner products. We have

$$\|\phi(z_1) - \phi(z_2)\|^2 = \|z_1 - z_2\|^2 \tag{4.2}$$

Note that for any $z \in H$, $\|\phi(z) - \phi(0)\|^2 = \|z - 0\|^2$ implies

$$\|\phi(z)\|^2 = \|z\|^2 \tag{4.3}$$

Expanding Equation 4.2, we have $\|\phi(z_1)\|^2 - 2\text{Re}\langle\phi(z_1), \phi(z_2)\rangle + \|\phi(z_2)\|^2 = \|z_1\|^2 - 2\text{Re}\langle z_1, z_2\rangle + \|z_2\|^2$ which implies $\text{Re}\langle\phi(z_1), \phi(z_2)\rangle = \text{Re}\langle z_1, z_2\rangle$.

Replacing z_2 with iz_2, we have $\text{Re}\langle\phi(z_1), \phi(iz_2)\rangle = \text{Re}\langle z_1, iz_2\rangle$. Using the identity $\text{Re}\langle x, iy\rangle \equiv \text{Im}\langle x, y\rangle$ and linearity of ϕ, we have $\text{Im}\langle\phi(z_1), \phi(z_2)\rangle = \text{Im}\langle z_1, z_2\rangle$. Thus we have shown that $\langle\phi(z_1), \phi(z_2)\rangle = \langle z_1, z_2\rangle$ for all $z_1, z_2 \in H$.

In particular, $\langle\phi(x_i), \phi(x_j)\rangle = \langle x_i, x_j\rangle = \delta_{ij}$. Thus, we have that $\{\phi(x_j)\}$ is orthonormal. Since ϕ is onto, $\{\phi(x_j)\}$ is a basis, thus it is an orthonormal basis. Thus setting $\phi(x_j) := y_j$, the isometry ϕ can be obtained in the fashion stated in the question.